Lectures on Hyperbolic Coxeter Groups

by

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The present Notes are based on a seminar given in 1967 at the University of Notre Dame. The purpose was to cover the main results related with the canonical linear representation of Coxeter groups and the realizations of Coxeter groups as properly discontinuous groups generated by reflections. Beside an algebraic introduction the main topics are: 1) the geometric properties of the canonical representation (results of Coxeter and Tits); 2) the linear representations close to the canonical representation (examples of Vinberg and Katz); 3) the determination and the properties of hyperbolic Coxeter groups (large part of the material in this section comes from F. Lanner and N. Bourbaki); 4) some results of Vinberg on arithmetic and non arithmetic hyperbolic Coxeter groups.

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Chapter IV. Arithmetic Hyperbolic Coxeter Groups

Chapter I. Coxeter Groups

§1. Groups Generated by Reflections.

Let E be a finite-dimensional real vector space and E* the dual space of E. We denote by < , > the pairing between E and E*. A reflection of E is a linear map $r:E \to E$ defined by the formula

$$r(x) = x - 2<x,f>e \text{ for any } x \in E,$$

where e and f are elements in E and E* respectively such that $<e,f> = 1$.

Let M be a connected differentiable manifold. By a reflection of M, we mean a diffeomorphism r of M such that

(i) r^2 = the identity;

(ii) the set $\{x \in M | r(x) \neq x\}$ is not connected.

(It is easy to see that this set has just two connected components.)

Let W be a group of diffeomorphisms of a simply connected manifold M, acting properly* on M and generated by a finite set of reflections. Then the following result is known.

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* This means that the map $\varphi:W \times M \to M \times M$ defined by $\varphi(w,x) = (wx,x)$ is a proper map, i.e. the inverse image $\varphi^{-1}(K)$ of any compact set K in $M \times M$ is compact.

[J. L. Koszul, Lectures on Transformation Groups. Tata Institute of Fundamental Research, 1964.]

Theorem 1-1. *There exists a finite set* T *in* W *satisfying the following conditions*:

(a) *Every element in* T *is a reflection of* M;

(b) *For any* t *and* t' *in* T *let* $p_{tt'}$ *denote the order of* tt' *in* W. *Then the group* W *is presented by the set of generators* T *and the set of relations* $(tt')^{P_{tt'}} = 1 \quad (t,\ t' \in T,\ P_{tt'} < \infty)$.

Remark 1-1. The condition (b) means the following: Let $F(T)$ be the free group generated by the finite set T. Denote by $\rho : F(T) \rightarrow W$ the natural homomorphism of $F(T)$ onto W. Then the kernel of π is the invariant subgroup generated by the $(tt')^{P_{tt'}}$ in $F(T)$ $(t,\ t' \in T,\ P_{tt'} < \infty)$.

§2. *Coxeter matrices and Coxeter groups*.

Let W be a group and S a set of elements of order 2 in W. Let $N = \{1,2,\ldots,\infty\}$ be the set of all the positive integers and the symbol ∞. For any $(s,t) \in S \times S$, let $p_{st} \in N$ be the order of st. Then

(i) $p_{st} = p_{ts}$ for $t \in S$ and $s \in S$

(ii) $p_{ss} = 1$ for $s \in S$

(iii) $p_{st} \geq 1$ if $s \neq t$.

Definition 2.1. *A Coxeter system (or Coxeter group) is a group* W *with a set* $S \subset W$ *satisfying the conditions*:

a) *each* $s \in S$ *is of order* 2,

b) S *is a set of generators for* W,

c) *if* p_{st} *denotes the order of* st, *then* W *is presented by the set of generators* S *and the set of relations* $(st)^{p_{st}} = e$ $(s, t \in S, p_{st} < \infty)$.

A matrix (p_{st}) with indices in a finite set S and with entries in N is called a *Coxeter matrix* if it satisfies the conditions (i), (ii), (iii). For any Coxeter system $\mathfrak{M} = \{W,S\}$ the matrix $(p^{\mathfrak{M}}_{st})$ with indices in S where $p^{\mathfrak{M}}_{st}$ is the order of st is a Coxeter matrix called the *matrix of the Coxeter system* $\{W,S\}$. From b) and c) in Definition 2.1 it follows that W is determined up to an isomorphism, by the Coxeter matrix $(p^{\mathfrak{M}}_{st})$.

We shall prove in Chapter II that *any Coxeter matrix is the matrix of a Coxeter system*.

Example 2-1. To any discrete group of diffeomorphisms of a simply connected manifold, acting properly and generated by reflections corresponds a Coxeter system.

Example 2-2. Let W be a group generated by two elements s, t of order 2. Then {W, s, t} is a Coxeter system; W is a dihedral group.

§3. Length and Reduced Expressions.

Let W be a group and $S \subset W$ a set of generators of order 2. Every element $w \in W$ may be written in the form

$$(3\text{-}1) \qquad w = s_1 \cdot s_2 \cdot \ldots \cdot s_r, \ s_j \in S, \ 0 \leq r.$$

By a reduced expression of w we mean a decomposition (3-1) of w with the minimal number of s_j's. The length $\ell(w)$ of w with respect to S is the number of the factors s_j in a reduced expression of w. Note that $\ell(w) = 0$ means $w = e$ and $\ell(w) = 1$ means $w \in S$.

Proposition 3-1. For any $w, w' \in W$

$$\ell(w^{-1}) = \ell(w)$$

$$\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$$

Proof. If $w = s_1 s_2 \ldots s_r$ is a reduced expression of w, then $w^{-1} = s_r s_{r-1} \dots s_1$, hence $\ell(w^{-1}) \leq \ell(w)$ and therefore $\ell(w^{-1}) = \ell(w)$. If $w' = s'_1 s'_2 \dots s'_s$ is a reduced expression of w', then $ww' = s_1 s_2 \dots s_r s'_1 s'_2 \dots s'_s$, hence $\ell(ww') \leq r + s = \ell(w) + \ell(w')$. On the other hand, $\ell(w) = \ell(ww'(w')^{-1}) \leq \ell(ww') + \ell(w')$.

From now on we denote by $\{W,S\}$ a Coxeter system. The length $\ell(w)$ of an element in W will always be the length with respect to the set of generators S.

Proposition 3.2. Let $\{W,S\}$ be a Coxeter system and let $\Delta = \{1,-1\}$ be the multiplicative group of order 2. There exists a homomorphism φ of W onto Δ such that $\varphi(S) =-1$ for all $s \in S$.

Proof. Let F(S) be the free group generated by the set S. Let ψ be the homomorphism of F(S) onto Δ such that $\psi(s) = -1$ for all $s \in S$. The kernel of ψ contains all elements of the form ss' with $s \in S$ and $s' \in S$. Therefore it contains the kernel of the homomorphism $\rho: F(S) \to W$ which extend the natural injection $S \to W$. Since ρ is surjectiv, there exists a homomorphism $\varphi: W \to \Delta$ such that $\psi = \varphi \circ \rho$. We have $\varphi(S) = \varphi(\rho(S)) = \psi(S) = -1$ for every $s \in S$.

Corollary 3-1. Let $\{W, S\}$ be a Coxeter system. For *any* $w \in W$ *and any* $w' \in W$, $\ell(ww') = \ell(w) + \ell(w') \bmod 2$.

Corollary 3-2. *Let* $\{W,S\}$ *be a Coxeter system. For any* $w \in W$ *and any* $s \in S$, $\ell(ws)$ *is either equal to* $\ell(w) + 1$ *or to* $\ell(w) - 1$.

Now we are going to prove the following theorem.

Theorem 3-1. *(Matsumoto [7]). Let* $\{W,S\}$ *be a Coxeter system and* $w \in W$ *and* $s \in S$. *Let* $w = s_1 \ldots s_p$ *be a reduced expression of* w. *If* $\ell(ws) < \ell(w)$, *then there exists a unique integer* $j \in [1,p]$ *such that* $ws = s_1 \ldots \hat{s}_j \ldots s_p$.

Let $F(S)$ be the free group generated by the set S and ρ the natural homomorphism from $F(S)$ onto W. For every $s \in S$, let $\underline{s}$ be the same element regarded as an element of $F(S)$, so that $\rho(\underline{s}) = s$. Let M be the subset of $F(S)$ consisting the identity e and all elements of the form $\underline{s}_1\underline{s}_2 \ldots \underline{s}_q$ with $s_i \in S$ and $q > 0$. Let X be the set of elements in W conjugate to an element in S. Define a homomorphism A from $F(S)$ into the group of permutations $P(X)$ of X by the formula

$$A(u)x = \rho(u)x\ \rho(u^{-1})$$

for every $u \in F(S)$ and every $x \in X$. Let $m = \underline{s}_1\underline{s}_2 \ldots \underline{s}_p$ be any element in M and x be any element in X. We denote by $h(m,x)$ the number of indices $i \in [1,p]$ such that $s_i = A(\underline{s}_{i+1} \ldots \underline{s}_p)x$. In the case $i = p$ the above equation means that $s_p = x$. In other words, $h(\underline{s}_1\underline{s}_2 \ldots \underline{s}_p, x)$ is the number of indices $i \in [1,p]$ such that $x = s_p\, s_{p-1} \ldots s_{i+1}\, s_i\, s_{i+1} \ldots s_{p-1}\, s_p$.

Lemma 3-1. *For any* m, m' *in* M, *we have*

$$h(mm', x) = h(m, A(m')x) + h(m', x).$$

This follows immediately from the definition of h.

Lemma 3-2. *Let* $(p_{ss'})$ *be the matrix of* $\{W,S\}$. *For all* $s, s' \in S$ *such that* $p_{ss'} < \infty$ *and for all* $x \in X$, *we have* $h((\underline{ss}')^{p_{ss'}}, x) \equiv 0 \bmod 2$.

Proof. For each $k \in [1,p]$, set $s_{2k-1} = s$ and $s_{2k} = s'$ and set $p = p_{ss'}$. Then we have $(ss')^p = \underline{s}_1\underline{s}_2 \ldots \underline{s}_{2p}$. Put $b_{2p} = \underline{s}_{2p}$ and $b_j = \underline{s}_{2p} \ldots \underline{s}_{j+1}\, \underline{s}_j\, \underline{s}_{j+1} \ldots \underline{s}_{2p}$ for $j \in [1,2p -$ Then we have $b_{j-1} = \underline{s}_{2p} \ldots \underline{s}_{j-1}\, \underline{s}_j\, \underline{s}_{j-1} \ldots \underline{s}_{2p}$ $= (\underline{s}_{2p} \ldots \underline{s}_{j-1}\, \underline{s}_j\, \underline{s}_{j+1} \ldots \underline{s}_{2p})(\underline{ss}') = b_j(\underline{ss}')$, since $\underline{s}_j = \underline{s}_{j+2}$ for all j. Then we get $b_{j-p} = b_j((\underline{ss}')^p) = b_j$ for all j in $[p+1, 2p]$. It follows immediately that $h((\underline{ss}')^p, x$ is an even integer.

Lemma 3-3. $h(m,x) = h(m',x) \pmod 2$ *for all* $x \in X$ *and all* $m, m' \in M$ *such that* $\rho(m) = \rho(m')$.

Proof. Define an action of the semi-group M on the product set $\Delta \times X$ by the formula

$$\underline{s} \cdot (a,x) = ((-1)^{h(\underline{s},x)}a,\ A(\underline{s}) \cdot x)$$

for any $s \in S$. This is well defined, since from Lemma 3-1 we can easily check that

$$m \cdot (a,x) = ((-1)^{h(m,x)}a,\ A(m) \cdot x)$$

for all $m \in M$. Now the elements of form $(\underline{ss}')^{P_{ss'}}$ $(P_{ss'} < \infty)$ induce the identity permutation on $\Delta \times X$, since $h((\underline{ss}')^{P_{ss'}}, x) \equiv 0 \pmod 2$ by Lemma 3-2 and $\rho((ss')^{P_{ss'}}) = e$. Therefore we may define the action of W on $\Delta \times X$ by the formula

$$\rho(u) \cdot (a,x) = u \cdot (a,x) \quad (u \in M)$$

Therefore $m \cdot (a,x) = m' \cdot (a,x)$ for any $m, m' \in M$ such that $\rho(m) = \rho(m')$, which implies $h(m,x) \equiv h(m',x) \pmod 2$.

Lemma 3-4. If the length of $s_1 \ldots s_p$ is p, then $h(s_1 \ldots s_p, x)$ is equal either to 0 or to 1 for any $x \in X$.

Proof. Put $b_p = s_p$ and $b_j = s_p \ldots s_{j+1} s_j s_{j+1} \ldots s_p$ for any j in [1, p-1]. Suppose now that $h(s_1 \ldots s_p, x) \geq 2$.

Then there exist indices i, j $(1 \leq i < j \leq p)$ such that $b_i = b_j$. Then we get $s_j\, s_{j-1} \dots s_{i+1}\, s_i\, s_{i+1} \dots s_{j-1} = e$, and hence $s_i\, s_{i+1} \dots s_{j-1} = s_{i+1} \dots s_j$. Then we have $s_1 \dots s_p = (s_1 \dots s_{i-1})(s_i \dots s_{j-1})(s_j \dots s_p)$ $= (s_1 \dots s_{i-1})(s_{i+1} \dots s_{j-1})(s_j \dots s_p)$, which implies that the length of $s_1 \dots s_p$ is less than p and this is a contradiction.

Proof of Theorem 3-1. Let $w = s_1 \dots s_p$ $(p = \ell(w))$ and assume $\ell(ws) = p - 1$. Let $ws = s_1'\, s_2' \dots s_{p-1}'$ $(s_j' \in S)$ be a reduced expression of ws. Then we have $w = s_1 \dots s_p =$ $s_1' \dots s_{p-1}'\, s$. By Lemma 3-3, we have $h(\underline{s}_1 \dots \underline{s}_p, s) =$ $h(\underline{s}_1' \dots \underline{s}_{p-1}'\, \underline{s}, s)$ (mod 2). By Lemma 3-4, $h(\underline{s}_1' \dots \underline{s}_{p-1}'\, \underline{s}, s)$ is either 0 or 1. But clearly $h(\underline{s}_1' \dots \underline{s}_{p-1}'\, \underline{s}, s) \geq 1$ and hence $h(\underline{s}_1 \dots \underline{s}_p, S) = 1$. This shows that there exists an integer j in $[1,p]$ such that $s_p \dots s_{j+1}\, s_j\, s_{j+1} \dots s_p$. Then we have $s_i\, s_{i+1} \dots s_p\, s = s_{i+1} \dots s_p$. Thus we get ws $= s_1 \dots \hat{s}_j \dots s_p$. The uniqueness assertion follows immediately from Lemma 3-4.

Corollary 3-3. *If* $w = s_1 \dots s_p$ *is of length* $< p$, *there exist integers* i, j $(1 \leq i < j \leq p)$ *such that* $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_p$.

Proof. Let p_0 be an integer in $[1,p]$ for which we have

$\ell(s_1 \ldots s_{p_0}) = p_0$ and $\ell(s_1 \ldots s_{p_0+1}) < p_0 + 1$. Then by Theorem 3-1, there exists an integer i $(1 \leq i \leq p_0)$ such that $s_1 \ldots s_{p_0+1} = s_1 \ldots \hat{s}_i \ldots s_{p_0}$ and thus $s_1 \ldots s_p = s_1 \ldots \hat{s}_i \ldots \hat{s}_{p_0+1} \ldots s_p$.

We now prove the converse of Theorem 3-1. Let G be a group generated by a finite set S of elements of order 2. We can define the length of an element of G. If the assertion* of Theorem 3-1 is satisfied, we say that the group G satisfies the _EX-condition_.

Theorem 3-2. _(Matsumoto [7])_. _Let_ G _be a group generated by a finite set_ S _of elements of order_ 2. _If the group_ G _satisfies the EX-condition, then_ {G,S} _is a Coxeter system_.

Theorem 3-2 follows easily from the following proposition.

Proposition 3-3. _Let_ G _be a group generated by a finite set_ S _of elements of order_ 2. _Suppose there exists an invariant subgroup_ K _of_ G _contained in_ G - S _such that the quotient group_ G/K _satisfies the EX-condition with respect to the image of_ S _in_ G/K. _Then_ K _is generated by the_

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* The uniqueness assertion is not necessary since it follows from the rest of the assertion.

elements of form $(ss')^{p_{ss'}}$ $(s, s' \in S, p_{ss'} < \infty)$, *where* $p_{ss'}$ *is the order of the coset in* G/K *containing* ss'.

The proof of Proposition 3-3 is divided into several lemmas.

Lemma 3-5. *The length of any element in* K *is even.*

Proof. Let $w = s_1 \ldots s_r$ be a reduced expression of any element w in K. Let $\pi : G \rightarrow G/K$ be the natural homomorphism. Then we have $\pi(w) = \pi(s_1)\ \pi(s_2)\ldots\pi(s_r) = e$. By the EX-condition on G/K, r must be even since the length of $e = \pi(s_1)\ldots\pi(s_r)$ is zero, (see Corollary 3-3).

Lemma 3-6. *Assume* K *is not trivial and let* $n = 2p$ *be the minimum of the lengths of the elements in* $K - \{e\}$. *If an element* $s_1 \ldots s_{p+1}$ *is of length* $p + 1$ *and* $\pi(s_1 \ldots s_{p+1})$ *is of length* $< p + 1$, *then* $s_2 \ldots s_p\ s_{p+1}\ s_p$ *is of length* $p + 1$ *and* $\pi(s_2 \ldots s_{p+1}\ s_p)$ *is of length* $< p + 1$. *Furthermore* $s_2 \ldots s_p\ s_{p+1}\ s_p \ldots s_1$ *belongs to* K.

Proof. By the EX-condition on G/K there exist integers i, j $(1 \leq i < j \leq p + 1)$ such that $\pi(s_1 \ldots s_{p+1}) = \pi(s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_{p+1})$. Hence $s_1 \ldots s_{p+1}\ s_{p+1} \ldots \hat{s}_j \ldots \hat{s}_i \ldots s_1$ belongs to K. Since any element in S is of order 2 and since K is invariant, $s_i \ldots s_j \ldots s_{j-1} \ldots s_{i+1}$ belongs to K. We will show that $i = 1$ and $j = p + 1$. In fact, if this were not the case, the length of $s_i \dots s_j \; s_{j-1} \dots s_{i+1}$ would be strictly less than $n = 2p$. From the definition of the number $n = 2p$, we have $s_i \dots s_j \; s_{j-1} \dots s_{i+1} = e$. Then $s_i \dots s_j = s_{i+1} \dots s_{j-1}$ and this is impossible since $s_i \dots s_{p+1}$ is a reduced expression. Thus $i = 1$ and $j = p + 1$ and hence $s_1 \dots s_{p+1} \; s_p \dots s_2$ belongs to K. Since K is invariant, $s_2 \dots s_{p+1} \; s_p \dots s_2 \; s_1$ belongs to K, which proves the last assertion. By the same argument, we can see that the length of $s_2 \dots s_p \; s_{p+1} \; s_p$ is $p + 1$. The other assertions follow from $\pi(s_2 \dots s_{p+1} \; s_p) = \pi(s_{p-1} \dots s_1)$.

Corollary 3-4. *In the notations of Lemma* 3-6, *we have* $(s_p \; s_{p+1})^p \in K - \{e\}$.

Proof. The element $s_2 \dots s_p \; s_{p+1} \; s_p$ also satisfies the conditions of Lemma 3-6. Applying Lemma 3-6 repeatedly, we find that eventually one of the two cases is possible.

(1) if $p \equiv 0 \; (2)$, $\overbrace{s_p \; s_{p+1} \dots s_{p+1} \; s_p}^{p+1}$ is of the length p +

and $\underbrace{s_p \; s_{p+1} \dots s_{p+1}}_{p+1} \; s_p \; \underbrace{s_{p+1} \dots s_p}_{p} \in K$,

(2) if $p \equiv 1(2)$, $\underbrace{s_p \; s_{p+1} \dots s_p \; s_{p+1}}_{p+1}$ is of the length p +

and $(s_p \; s_{p+1})^p \in K$.

Since K is invariant, the case (1) cannot occur. Since the length of $\underbrace{s_p\ s_{p+1} \cdots s_p}_{p+1}$ is p + 1, the length of $(s_p\ s_{p+1})^p$ is greater than 2. Hence $(s_p\ s_{p+1})^p \neq e$.

Proof of Proposition 3-3. Let K' be the invariant subgroup generated by all elements of the form $(ss')^{p_{ss'}}$ $(p_{ss'} < \infty)$. Obviously, K' is contained in K. We denote by $\overline{G}$ (resp. $\overline{K}$) the quotient group G/K' (resp. K/K'). We also denote by $\overline{s}$ the coset of $s \in S$ in G/K'. Then $\overline{G}$ and $\overline{K}$ also satisfy the conditions in Proposition 3-3. Suppose $\overline{K}$ is not trivial. Let $\overline{w} = \overline{s}_1 \ldots \overline{s}_{2p}$ be an element of K - {identity} with the minimal length. Then $\overline{s}_1 \ldots \overline{s}_p\ \overline{s}_{p+1}$ satisfies the conditions of the Lemma 3-6. By Corollary 3-2, we have $(s_{\overline{p}}\ s_{\overline{p+1}})^{\overline{p}}$ $\in \overline{K}$ - {identity}. But this is impossible since $(s_{\overline{p}}\ s_{\overline{p+1}})^{\overline{p}}$ is the identity, by the definition of K'.

Corollary 3-5. Let {W,S} be a Coxeter system. Let S' be a subset of S and W' the subgroup of W generated by S'. Then {W',S'} is a Coxeter system.

Proof. Since the EX-condition holds in W, for any $w \in W'$ the length of w as an element of W with respect to S equal to the length of w as an element of W' (with respect to S'). We can then apply Theorem 3-2 to {W',S'}.

§4. The Graph of a Coxeter Group

Let $(P_{ss'})$ be a Coxeter matrix with indices in a finite set S. The graph of the Coxeter matrix $(P_{ss'})$ is by definition a 1-dimensional simplicial complex $\mathcal{G}(P_{ss'})$ with vertices $\{v_s\}_{s\in S}$ such that two vertices v_s and v_t are the boundary of a 1-simplex in $\mathcal{G}(P_{ss'})$ if and only if $P_{ss'} \geq 3$.

When the graph of a Coxeter matrix is connected, the Coxeter matrix is called irreducible. Let $\mathcal{W} = \{W,S\}$ be a Coxeter system. The graph $\mathcal{G}(\mathcal{W})$ of $\mathcal{W}$ is by definition that of the Coxeter matrix $(P_{ss'})$. The Coxeter system is called irreducible if the Coxeter matrix $(P_{ss'})$ is irreducible.

Remark 4-1. When $\mathcal{G}$ is not connected, let $\{\mathcal{G}_\alpha\}$ be the connected components of $\mathcal{G}$. If S_α denotes the set of elements $s \in S$ corresponding to a vertex in $\mathcal{G}_\alpha$, then W is the direct product of the subgroups W_α generated by S_α. For each α, $\mathcal{G}_\alpha$ is the graph of the Coxeter system $\{W_\alpha, S_\alpha\}$.

§5. The Canonical Bilinear Form of a Coxeter Group.

Let $(P_{ss'})$ be a Coxeter matrix with indices in a finite set S. Denote by $\{e_s\}_{s\in S}$ the natural basis of $\mathbb{R}^S$. We define a bilinear form B on $\mathbb{R}^S$ as follows:

$$B(e_s, e_{s'}) = \begin{cases} -\cos\dfrac{\pi}{p_{ss'}} & \text{if } p_{ss'} < \infty \\ -1 & \text{if } p_{ss'} = \infty. \end{cases}$$

The bilinear form B is called the canonical bilinear form of the Coxeter matrix $(P_{ss'})$. In particular $B(e_s, e_s) = 1$ for each $s \in S$.

Let $\mathcal{W} = \{W,S\}$ be a Coxeter system. The canonical bilinear form B of the matrix of $\mathcal{W}$ is called the canonical bilinear form of $\mathcal{W}$.

Chapter II

Linear Representations of Coxeter Groups

§6. The Canonical Representation of a Coxeter Group.

Let S be a finite set and $(p_{ss'})$ be a Coxeter matrix with indices in S. Let F(S) be the free group generated by S. For any $s \in S$, we define a reflection $\rho(s)$ of $\mathbb{R}^S = E$ as follows:

$$\rho(s)x = x - 2B(x,e_s)e_s \text{ for any } x \in E.$$

<u>Lemma</u> 6-1. <u>For any</u> $x \in S$, <u>the canonical bilinear form</u> B <u>is invariant under</u> $\rho(s)$.

In fact, for any $s \in S$ and any $x \in E$,

$$B(\rho(s)x,\ \rho(s)x) = B(x - 2B(x,e_s)e_s,\ x - 2B(x,e_s)e_s)$$

$$= B(x,x) - 4B(x,e_s)^2 + 4B(x,e_s)^2\ B(e_s,e_s) = B(x,x).$$

<u>Lemma</u> 6-2. <u>For any</u> $s \in S$ <u>and</u> $s' \in S$, <u>the order of</u> $\rho(s)\ \rho(s')$ <u>is</u> $p_{ss'}$.

<u>Proof</u>. (As page 18).

<u>Lemma</u> 6-3. <u>Let</u> K <u>be the invariant subgroup in</u> F(S)

generated by the elements $(ss')^{p_{ss'}}$ $(s,\ s' \in S)$ *and let* q *be the canonical homomorphism of* $F(S)$ *onto* $F(S)/K$. *Then*

a) q *is injective on* S,

b) *for any* $s \in S$, $q(s)$ *is of order* 2,

c) $\mathfrak{WP} = \{F(S)/K,\ q(S)\}$ *is a Coxeter system,*

d) *the matrix* $p^{\mathfrak{WP}}_{q(s)q(s')}$ *of* $\mathfrak{WP}$ *is given by*

$$p^{\mathfrak{WP}}_{q(s)q(s')} = p_{ss'} \qquad (s,\ s' \in S).$$

Proof. Lemma 6-2 shows that there exists a linear representation σ of $F(S)/K$ in $\mathbb{R}^S$ such that $\sigma(s) = \rho(q(s))$ for any $s \in S$. Therefore the order of $q(s)q(s')$ is $p_{ss'}$. Since $s \neq s'$ implies $p_{ss'} > 1$, q is injective on S. Since $p_{ss} = 1$, $q(s)$ is of order 2. The statements c) and d) are now obvious.

Proposition 6-1. *Any Coxeter matrix is the matrix of a Coxeter system*.

Proof. Follows from Lemma 6-3.

Proposition 6-2. *Let* $\{W,S\}$ *be a Coxeter system and let* B *be the canonical bilinear form of* $\{W,S\}$. *There exists one and only one linear representation* ρ *of* W *in* $\mathbb{R}^S$ *such that*

$$\rho(s)x = x - 2B(x,e_s)e_s$$

for $s \in S$, $x \in \mathbb{R}^S$.

Proof. The existence follows from Lemma 6-2. Since W is generated by S, uniqueness holds.

The linear representation ρ of W in $\mathbb{R}^S$ is called the canonical representation of the Coxeter group W. The canonical bilinear form is invariant under this representation.

Proof of Lemma 6-2. It is clear that $(\rho(s))^2$ is the identity. Therefore we may assume $s \neq s'$. Since $B(e_s, e_{s'}) = -\cos\frac{\pi}{p_{ss'}} < 1$, the restriction of B to the two-dimensional subspace $F = e_s + e_{s'}$ of E is positive definite. Put $F' = \{x \in E | B(x,F) = 0\}$. Then we have $E = F \oplus F'$ (direct sum). It is easy to see that both F and F' are invariant under $\rho(s)$ and $\rho(s')$. Since both $\rho(s)$ and $\rho(s')$ act on F' as the identity transformation, we need only consider the restriction of $(\rho(s)\ \rho(s'))$ to F. We have $\rho(s')\ \rho(s)e_s = -e_s - 2\cos\frac{\pi}{p_{ss'}}\ e_s$ Hence $B(\rho(s')\ \rho(s)e_s,\ e_s) = -1 + 2\cos^2\frac{\pi}{p_{ss}} = \cos\frac{2\pi}{p_{ss'}}$. This shows that $(\rho(s)\ \rho(s'))^{p_{ss'}}$ is the identity transformation of E, and that the order of $\rho(s)\ \rho(s')$ is precisely $p_{ss'}$.

§7. Properties of the Canonical Representation.

Let $\mathcal{W} = \{W,S\}$ be an irreducible Coxeter system, B its canonical bilinear form and $\rho : W \to GL(E)$ the canonical representation.

Proposition 7-1. *Any invariant subspace* $F(\neq E)$ *of the canonical representation* ρ *is orthogonal to* E *with respect to the canonical bilinear form* B.

Proof. We first show that F contains no element of the basis $\{e_s \ (s \in S)\}$. In fact, let $S' = \{s \in S \mid e_s \in F\}$ and assume S' is not empty. Since $\mathcal{W}$ is irreducible and $F \neq E$, there exist $e_t \in S'$ and $e_{t'} \in S - S'$ such that $B(e_t, e_{t'}) \neq 0$. Then $\rho(t')e_t = e_t - 2B(e_t, e_{t'})e_{t'}$, i.e. $2B(e_t, e_{t'})e_{t'} = e_t - \rho(t')e_t$. Since F is an invariant subspace, $e_t - \rho(t')e_t$ is in F, so that $e_{t'}$ must be in F. This is a contradiction. Now, take any $x \in F$. For any s in S, we have

$$x - \rho(s)x = 2B(x,e_s)e_s, \text{ i.e. } 2B(x,e_s)e_s \in F$$

and therefore $B(x,e_s) = 0$ for all $s \in \mathcal{S}$.

Proposition 7-2. *If the canonical bilinear form* B *is degenerate, then the canonical representation* ρ *is not* *completely reducible*.

Proof. Define E^o by $\{x \in E | B(x,E) = 0\}$. Since B is invariant under $\rho(W)$, E^o is an invariant subspace of ρ. Since $B(e_s, e_s) = 1$ for all $s \in S$, and since B is degenerate, E^o is a non-trivial subspace. Suppose ρ is completely reducible, then there exists a non-trivial invariant subspace F such that

$$E = E^o \oplus F .$$

But $F \subset E^o$ by Proposition 7-1. This is a contradiction. Therefore ρ is not completely reducible.

Corollary 7-1. *If the canonical bilinear form* B *is degenerate*, W *is an infinite group*.

Theorem 7-1. (Witt). *Let* $\mathcal{W} = \{W,S\}$ *be a Coxeter system. The group* W *is finite if and only if its canonical bilinear form* B *is positive definite*.

Proof. See [11].

§8. *The Fundamental Cone*.

We consider the dual representation (denoted by ρ^*) of the canonical representation $\rho:W \to GL(E)$ of a Coxeter group. For each $s \in S$ we define an element $e_s^B \in E^*$ by $<y, e_s^B> = B(y, e_s)$. Then we have

$$\begin{cases} \rho^*(s)x = x - 2<x,e_s>e_s^B \\ \rho(s)y = y - 2<y,e_s^B>e_s \end{cases} \qquad (s \in S,\ y \in E,\ x \in E^*)$$

We write wx instead of $\rho^*(w)x$ for $w \in W$ and $x \in E^*$.

For any $s \in S$, we define A_s (resp. H_s) by

$$A_s = \{x \in E^* | <e_s, x> > 0\} \text{ (resp. } H_s = \{x \in E^* | <e_s, x> = 0\}).$$

Then $E = A_s \cup H_s \cup sA_s$ (disjoint union) and s is the identity transformation on H_s. Now we define a cone C in E^* by

$$C = \bigcap_{s \in S} A_s$$

and call C the fundamental cone of the Coxeter group W. Then

$$\overline{C} = \{x \in E^* | <e_s,x> \geq 0 \text{ for all } s \text{ in } S\}.$$

Lemma 8-1. For any $s, s' \in S$ $(s \neq s')$ and $w \in W_{\{s,s'\}}$ (the subgroup of W generated by s and s') either

(i) $w(A_s \cap A_{s'}) \subset A_s$ and $\ell(sw) = \ell(w) + 1$,

or

(ii) $w(A_s \cap A_{s'}) \subset sA_s$ and $\ell(sw) = \ell(w) - 1$.

Proof. It is easy to prove the lemma when $W = W_{\{s,s'\}}$. But the general problem may be reduced to this cas

Lemma 8-2. For any $s \in S$ and $w \in W$, either

(1) $wC \subset A_s$ and $\ell(sw) = \ell(w) + 1$,

or

(2) $wC \subset sA_s$ and $\ell(sw) = \ell(w) - 1$.

Proof. Let (P_n) and (Q_n) $(n \in N)$ be the following assertions.

(P_n) for any $s \in S$ and $w \in W$ of length n, either

(i) $wC \subset A_s$,

or

(ii) $wC \subset sA_s$ and $\ell(sw) = \ell(w) - 1$,

(Q_n) for any $s, s' \in S$ $(s \neq s')$ and $w \in W$ of length n, there exists a $u \in W_{ss'}$ such that $wC \subset u(A_s \cap A_{s'})$ and $\ell(w) = \ell(u) + \ell(u^{-1}w)$.

[I]. We first show that (P_n) and (Q_n) together imply (P_{n+1}). In fact, take w of length n + 1. We may write $w = s'w'$ $(s' \in S, \ell(w') = n)$.

(a) If $s' = s$, apply (P_n) to w' and s.

Since $\ell(sw') = \ell(w) > \ell(w')$, the second case in (P_n) does not occur. Therefore $wC = sw'C \subset sA_s$ and $\ell(sw) = \ell(w') = \ell(w) - 1$. This is the second case in (P_{n+1})

(b) If $s' \neq s$, apply (Q_n) to w', s' and s. Then there exists $u \in W_{ss'}$ such that $w'C \subset u(A_s \cap A_{s'})$, and $\ell(w') = \ell(u) + \ell(u^{-1}w')$. Hence $wC \subset s'w'C \subset s'u(A_s \cap A_{s'})$. Since $s'u \in W_{ss'}$, we apply Lemma 8-1 to $s'u$, s' and s. Then either

(i) $s'u(A_s \cap A_{s'}) \subset A_s$, and $\ell(ss'u) = \ell(s'u) + 1$,

or

(ii) $s'u(A_s \cap A_{s'}) \subset sA_s$ and $\ell(ss'u) = \ell(s'u) - 1$.

When the case (i) occurs, we have $wC \subset s'u(A_s \cap A_{s'}) \subset A_s$, implying the first case in (P_{n+1}). When (ii) occurs, we have $wC \subset s'u(A_s \cap A_{s'}) \subset sA_s$ and $\ell(sw) = \ell(ss'w') = \ell(ss'uu^{-1}w') \leq \ell(ss'u) + \ell(u^{-1}w') = \ell(sIu) - 1 + \ell(w') - \ell(u) \leq \ell(w') < \ell(w)$ (the equality in the middle comes from $\ell(w') = \ell(u) + \ell(u^{-1}w')$ and $\ell(ss'u) = \ell(s'u) - 1$). This is the second case in (P_{n+1}). Thus we have proved that (P_n) and (Q_n) imply (P_{n+1}).

[II]. We now prove that (P_{n+1}) and (Q_n) imply (Q_{n+1}). In fact, take $w \in W$ of length $n + 1$. Assume that

(a) $wC \subset A_s$ and $wC \subset A_{s'}$.

Then (Q_{n+1}) holds, with $u = e$. When (a) is not the case, we may assume (by (P_{n+1})) $wC \subset sA_s$ and $\ell(sw) = \ell(w) - 1$. Let $w' = sw$. Then the length of w' is n. Applying (Q_n) to w', s and s', there exists u in $W_{ss'}$ such that $swC \subset u\ (A_s \cap A_{s'})$ and $\ell(sw) = \ell(u) + \ell(u^{-1}sw)$. Hence $wC \subset su(A_s \cap A_{s'})$. Since $su \in W_{ss'}$ we have only to show that $\ell(w) = \ell(u) + \ell(u^{-1}w)$. We have $\ell(w) = \ell(sw) + 1 = \ell(u) + \ell(u^{-1}sw) + 1 \geq \ell(su) + \ell(u^{-1}sw) \geq \ell(w)$, so that $\ell(w) = \ell(su) + \ell(u^{-1}sw)$. Thus we have proved that (P_{n+1}) and (Q_n) imply (Q_{n+1}).

Now the assertions (P_1) and (Q_1) hold. Therefore we have proved Lemma 8-2 except for the last assertion in case (1). To prove that $\ell(sw) = \ell(w) + 1$ in case (1), suppose $wC \subset A_s$. Then we have $swC \subset sA_s$, which implies $\ell(s \cdot sw) = \ell(sw) - 1$ and thus $\ell(sw) = \ell(w) + 1$.

Theorem 8-1. If $wC \cap C \neq \phi$ $(w \in W)$, then w is the identity.

Proof. Suppose $wC \cap C \neq \phi$ and w were not the identity. Let $w = s_1 \ldots s_r$ be a reduced expression for w. Lemma 8-2 shows that $wC \subset A_{s_1}$ and $\ell(s_1 w) = \ell(w) + 1$. In particular we have $\ell(w) = \ell(s_2 \ldots s_r) - 1$, which is impossible.

Corollary 8-1. The representations ρ and ρ^* are faithful.

§9. The Faces of The Fundamental Cone and The Main Lemma.

For any subset X of S, we define a simplex C_X by

$$C_X = (\bigcap_{s \in X} H_s) \cap (\bigcap_{t \in S-X} A_t).$$

We remark $C_\phi = C$. The faces of C consist of C_X where X ranges over all subsets of S.

Main Lemma. Let w and w' be elements in W and X and X' subsets of S. If $wC_X \cap w'C_{X'} \neq \phi$, then $X = X'$ and $wW_X = w'W_X$, where W_X is the subgroup of W generated by $X \subset S$.

Proof. Without loss of generality, we may assume $w' = e$. We shall prove the lemma by induction on the length of w.

(i) When $\ell(w) = 0$ (i.e. $w = e$), the lemma is trivial since $C_X \cap C_{X'} \neq \phi$ clearly implies $X = X'$.

(ii) Suppose the lemma is true for all elements of length $< \ell(w)$ $(\ell(w) > 0)$. Since $\ell(w) > 0$, there exists an $s \in S'$ such that $\ell(sw) = \ell(w) - 1$. From Lemma 8-2, we have $wC \subset sA_s$; in particular we have $w\overline{C} \subset s\overline{A}_s$. On the other hand, we have

$$\phi \neq wC_X \cap C_{X'} \subset w\overline{C} \cap \overline{C} \subset s\overline{A}_s \cap \overline{A}_s = H_s.$$

Hence $C_{X'} \cap H_s \neq \phi$, which implies $s \in X'$. In particular, $C_{X'} \subset H_s$. Since s is the identity on H_s, we have $s(wC_X \cap C_{X'}) = swC_X \cap C_{X'}$. Since $swC_X \cap C_{X'} \neq \phi$ and $\ell(sw) < \ell(w)$, the hypothesis of the induction yields $x = x'$ and $swW_X = W_X$ and therefore $wW_X = sW_X = W_X$ (since $s \in X' = X$).

<u>Proposition</u> 9-1. <u>For any</u> $x \in C_X$, <u>the subgroup</u> W_X <u>is the isotropy subgroup of</u> W <u>at</u> x.

<u>Proof</u>. Since C_X is contained in $\bigcap_{s \in X} H_s$, $wx = x$ for any w in W_X. Conversely, if $wx = x$, then $wC_X \cap C_X \neq \phi$. From The Main Lemma, we get $w \in W_X$.

<u>Proposition</u> 9-2. <u>If</u> $wx \in \overline{C}$ <u>for some</u> x <u>in</u> C, <u>then</u> $x = wx$.

<u>Proof</u>. There exist X and X' ($X, X' \subset S$) such that $x \in C_X$ and $wx \in C_{X'}$. Since $wC_X \cap C_{X'}$ contains wx, we get $X = X'$ and $w \in W_X$ by the Main Lemma. Hence $wx = x$ by Proposition 9-1.

§10. <u>The Range Ω of a Coxeter Group</u>.

We denote by $\mathcal{F}$ the collection of all subsets in $W\overline{C}$

of the form wC_X for some w in W and some subset X of S. Then $W\overline{C}$ is the disjoint union of the sets in $\mathcal{F}$. We denote by Ω the interior of $W\overline{C}$ and call it the range of the Coxeter group W. Then clearly, $\Omega \supset C$ and Ω is stable under W.

Proposition 10-1. (Tits [8]). *The subset $W\overline{C}$ is convex. Hence the range Ω is also convex.*

We shall prove the following stronger result.

Lemma 10-1. *For any $x \in W\overline{C}$, $\operatorname{Conv}(x,\overline{C})$ is contained in the union of a finite number of sets of $\mathcal{F}$, where $\operatorname{Conv}(x,\overline{C})$ denotes the convex closure of $\{x,\overline{C}\}$.*

To prove this lemma, we will need Lemma 10-2. For any x in $W\overline{C} - \overline{C}$, we define the subset S^* of S by $S^* = \{s \in S | x \in sA_s\}$. Since $x \notin \overline{C}$, $S^* \neq \phi$.

Lemma 10-2. $\operatorname{Conv}(x,\overline{C}) \subset \overline{C} \cup [\bigcup_{s \in S^*} \operatorname{Conv}(x, \overline{C} \cap H_s)]$.

Proof. Let y be a point in $\overline{C}$. For any $s \in S$, we define $\varphi_s(r)$ for $r \in [0,1]$ by

$$\varphi_s(r) = \langle e_s, y + r(x - y)\rangle.$$

Since $y \in \overline{C}$, we have $\varphi_s(0) = \langle e_s, y\rangle \geq 0$. We define the real number θ by

$$\theta = \sup\{t \in [0,1] \mid \varphi_s(t) \geq 0 \text{ for any } s \text{ in } S\}.$$

Since $x \notin \overline{C}$, there exists an $s_0 \in S^*$ such that $\varphi_{s_0}(\theta) = 0$ and $\varphi_{s_0}(1) < 0$. Since $\varphi_{s_0}(1) < 0$, we have $x \in s_0 A_{s_0}$. Take $z = y + \theta(x - y)$. Then z is then in $\overline{C} \cap H_{s_0}$ and the segment $\{y + r(x - y) \mid r \in [0,1]\}$ is contained in $\overline{C} \cup \mathrm{Conv}(x, \overline{C} \cap H_{s_0})$. Furthermore we have seen that x is in $s_0 A_{s_0}$, which completes the proof of Lemma 10-2.

Proof of Lemma 10-1. Assume x to be contained in $w\overline{C}$. We shall prove this lemma by induction on $\ell(w)$.

(i) When $\ell(w) = 0$, the lemma is trivial.

(ii) We assume $\ell(w) > 0$ and that the lemma is valid for points in $w'\overline{C}$ with $\ell(w') < \ell(w)$. Since the lemma is trivial for $x \in \overline{C}$, we may assume $x \notin \overline{C}$. Now for any $s \in S$ we have $\mathrm{Conv}(x, \overline{C} \cap H_s) = s\cdot\mathrm{Conv}(sx, \overline{C} \cap H_s) \subset s\cdot\mathrm{Conv}(sx, \overline{C})$. From Lemma 8-2, we obtain $\ell(sw) = \ell(w) - 1$ for any $s \in S^*$. Therefore, by the assumption, $\mathrm{Conv}(sx, \overline{C})$ (in particular $\mathrm{Conv}(x, \overline{C} \cap H_s)$) is contained in a finite number of sets of . Then Lemma 10-1 follows from Lemma 10-2.

Proposition 10-1. *For any* $x \in \Omega$, *there exists an open cone* V *in* Ω *containing* x. *Furthermore we can take as* V *the interior of the union of a finite number of sets in* $\mathfrak{F}$.

Proof. Choose a $y \in C$. Since Ω is open, there exists a positive number ε such that $y + (1 + \varepsilon)(x - y) \in \Omega$. Then we can take as V the interior of $\mathrm{Conv}(z, \overline{C})$. The last assertion follows immediately from Lemma 10-1.

§11. *Properties of The Range Ω.*

Lemma 11-1. *Let* X *be a subset of* S *such that* W_X *is finite. Then we have* $W_X(\bigcap_{s \in X} \overline{A}_s) = E^*$.

Proof. Define L by

$$L = \mathop{}_{\substack{s \neq t \\ s,t \in S}} H_s \cap H_t .$$

and set $M = \bigcap_{s \in X} \overline{A}_s$. For any $x \in M - M \cap L$, $M \cup sM$ is a neighborhood of x for some $s \in X$. Hence the boundary of $W_X M$ is contained in $W_X L$. Since W_X is finite, the set $W_X L$ has codimension 2. Therefore $W_X M$ must be E^*.

Theorem 11-1. *The following three conditions are equivalent:*

(a) $C_X \subset \Omega$,

(b) $C_X \cap \Omega \neq \phi$,

(c) W_X *is finite*.

Proof. (b) $\Longrightarrow$ (c). We choose $x \in C_X \cap \Omega$ and fix it once and for all. Let w be any element in W_X. Since $x \in \overline{C}$, we have $wx \in w\overline{C}$. On the other hand, by Proposition 9-1, we have $wx = x$. Therefore $wC \cap U \neq \phi$ for any open neighborhood of x. By Proposition 10-1, there exists a cone $V \subset \Omega$ such that $x \in V$, which is the interior of the union of a finite number of sets $\{\nu\ C_Y\}$ of $\mathcal{F}$, i.e. $\overline{V} = \bigcup_{\nu} \nu\ C_Y$. Since $wC \cap V \neq \phi$, we have $wC \cap \nu_0\ C_{Y_0} \neq \phi$ for some $\nu_0 \in W$. From the Main Lemma, we have $Y_0 = \phi$ and $w = \nu_0$. Since the index set of ν is finite, we have proved that W_X is finite.

(c) $\Longrightarrow$ (a). Set $M = \bigcap_{s \in X} \overline{A}_s$ and $N = \bigcap_{s \notin X} \overline{A}_s$ and take $x \in C_X$. Clearly $x \in \bigcap_{s \notin X} A_s$. Therefore N is a neighborhood of x. Since W_X is finite and W_X is the isotropy group of W at x, $\bigcap_{w \in W_X} wN$ is also a neighborhood of x. Since $E^* = W_X M$ by Lemma 11-1, for any $y_0 \in \bigcap_{w \in W_X} wN$ there exists a $w_0 \in W_X$ such that $y_0 \in w_0 M$. On the other hand, $y_0 \in w_0 N$, so that $y_0 \in w_0 N \cap w_0 M = w_0 (\bigcap_{s \notin X} \overline{A}_s) \cap w_0 (\bigcap_{s \in X} \overline{A}_s) = w_0 \overline{C}$. Thus $x \in \bigcap_{w \in W_X} wN \subset W\overline{C}$. Therefore x is an interior point of $W\overline{C}$, i.e. $C_X \subset \Omega$.

We denote by $\mathcal{F}_0$ the sub-collection of $\mathcal{F}$ consisting of the sets wC_X for which W_x is finite.

Corollary 11-1. The range Ω is the disjoint union of the sets in $\mathcal{F}_0$.

Corollary 11-2. For any $x \in \Omega$, we can take as an open neighborhood of x in Ω the union of a finite number of sets in $\mathcal{F}_0$.

Proof. This is immediate by Theorem 11-1 and Proposition 10-1.

Theorem 11-2. W acts properly on the range Ω. In particular W is a discrete subgroup of GL(E*).

Let x and y be two points in Ω. And let V_x (resp. V_y) be an opne neighborhood of x (resp. y), as in Corollary 11-2. We also assume V_x (resp. V_y) is of the form $\bigcup_{\nu, x} \nu\, C_x$ (resp. $\bigcup_{\mu, y} \mu\, C_y$). We have only to show that the set $\{w \in W \mid wV_x \cap V_y \neq \phi\}$ is finite. In fact, this set is contained in the finite union of the sets $\{w \in W \mid w(\nu \cdot C_x) \cap (\mu \cdot C_y) \neq \phi\}$, where ν and μ (resp. X and Y) range over the finite subset of W (resp. $\mathcal{F}_0$). By the Main Lemma, $w(\nu \cdot C_x) \cap \mu \cdot C_y \neq \phi$ implies $w \nu W_x = \mu W_x$. Hence we have

$w \in \mu W_X \nu^{-1}$. Since $\mu W_X \nu^{-1}$ is finite, the set $\{w \in W | w(\nu C_X) \cap \mu C_Y \neq \phi\}$ is finite, and so is the set $\{w \in W | wV_x \cap V_y \neq \phi\}$.

Let PE^* be the real projective space associated with the real vector space E^*. We denote by $q: E^* - \{0\} \to PE^*$ the natural projection from $E^* - \{0\}$ onto PE^*. Then $GL(E^*)$ operates naturally on PE^* and q is an equivalent mapping. We denote by $P\Omega$ the image in PE^* of $\Omega - (0)$. Then W also acts on $P\Omega$. The following is obvious from the proof of Theorem 11-2.

Corollary 11-3. W *acts properly on* $P\Omega$.

Theorem 11-3. *The following three conditions are equivalent*:

(a) *the quotient space* $W \backslash P\Omega$ *is compact*,

(b) $\overline{C} - \{0\}$ *is contained in* Ω,

(c) *for any* $X \subsetneqq S$, W_X *is finite*.

Proof. We have already shown that (b) and (c) are equivalent. Now we shall prove that (b) implies (a). In fact, $q(\overline{C} - \{0\})$ is clearly compact in PE^*. Condition (b) implies $q(\overline{C} - \{0\}) \subset q(\Omega)$ and therefore $W \cdot q(\overline{C} - \{0\}) \subset q(\Omega)$. On the other hand $W \cdot q(\overline{C} - \{0\}) \supset q(W\overline{C} - \{0\}) \supset q(\Omega)$ and hence $W \cdot q(\overline{C} - \{0\}) = q(\Omega)$. This means that $W \backslash q(\Omega)$ is compact. Next we shall prove that (a) implies (b). In fact, since $W \backslash q(\Omega)$ is compact and since $q(\Omega)$ is locally compact, there exists a compact set K in $q(\Omega)$ such that $WK = q(\Omega)$. Now we know that for any point $x \in \Omega$ there exists a cone V_x, which is a neighborhood of x such that the set $\{w \in W | w\overline{C} \cap V_x \neq \phi\}$ is finite (see the proof of Theorem 11-2). It is easy to see the set $W_K = \{w \in W | wq(\overline{C} \cap \Omega) \cap K \neq \phi\}$ is finite. Since $wK \cap q(\overline{C} \cap \Omega) \neq \phi$ implies $w \in W_K^{-1}$, we see that $q(\overline{C} \cap \Omega) \subset W_K^{-1}K$. (Here we note that $WK = q(\Omega)$.) Since $W_K^{-1}K$ is compact in $q(\Omega)$ and $q(\overline{C} \cap \Omega)$ is closed in $q(\Omega)$, $q(\overline{C} \cap \Omega)$ is also closed in PE^*. Since $\overline{C} \cap \Omega$ is a cone, and since $q(\overline{C} \cap \Omega)$ is closed, $\overline{C} \cap \Omega$ is closed in $E^* - \{0\}$. Since C is in $\overline{C} \cap \Omega$, the closure of C in $E^* - \{0\}$ is contained in $\overline{C} \cap \Omega$. Hence we have $\overline{C} - \{0\} \subset \Omega$.

§12. Non Canonical Representations.

Let $\mathfrak{M} = \{W,S\}$ be a Coxeter system. Let $\Lambda = (\lambda_{st})$ be a square matrix with indices in S such that

(1) $\lambda_{st} > 0$ for any pair (s,t),

(2) $\lambda_{st}\,\lambda_{ts} = 1$ for any pair (s,t).

We set $a^{\Lambda}_{st} = -\lambda_{st} \cos \frac{\pi}{p_{st}}$ for any pair (s,t) and define a family of vectors $e^{\Lambda}_{s} \in E^*$ by the conditions $\langle e_t, e^{\Lambda}_{s} \rangle = a^{\Lambda}_{st}$ for any $t \in S$. For each $s \in S$, we define an endomorphism s_{Λ} of E by

$$s_{\Lambda}(x) = x - 2\langle x, e_s \rangle\, e^{\Lambda}_{s} \qquad \text{for any } x \in E.$$

Since $\langle e_s, e^{\Lambda}_{s} \rangle = a^{\Lambda}_{ss} = -\cos \frac{\pi}{p_{ss}} = 1$, the map s_{Λ} is a reflection for any $s \in S$.

<u>Lemma</u> 12-1. <u>For any</u> $s,t \in S$, $(s_{\Lambda} t_{\Lambda})^{p_{st}} = 1$ <u>provided</u> $p_{st} < \infty$.

<u>Proof</u>. We show that there exists a θ in $GL(E^*)$ such that

(1) $\theta(H_s \cap H_t) \subset H_s \cap H_t$,

(2) $s_{\Lambda} = \theta^{-1} s \theta \bmod (H_s \cap H_t)$,

(3) $t_{\Lambda} = \theta^{-1} t \theta \bmod (H_s \cap H_t)$.

In fact, let $\{e'_s\}_{s \in S}$ be the dual basis of E^* for the canonical basis $\{e_s\}_{s \in S}$ of E. With respect to this dual basis, s (resp. t) has the following matrix;

$$s = \left(\begin{array}{cc|c} -1 & 0 & \\ -2a_{st} & 1 & 0 \\ \hline \multicolumn{2}{c|}{\#} & I \end{array}\right)$$

$$t = \left(\begin{array}{cc|c} 1 & -2a_{ts} & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{\#'} & I \end{array}\right).$$

We define $\theta \in GL(E^*)$ by the matrix

$$\theta = \left(\begin{array}{cc|c} \sqrt{\lambda_{st}} & 0 & \\ 0 & \frac{1}{\sqrt{\lambda_{st}}} & 0 \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right).$$

Then

$$\theta^{-1}s\theta = \left(\begin{array}{cc|c} -1 & 0 & \\ -2\lambda_{st}a_{st} & 1 & 0 \\ \hline \multicolumn{2}{c|}{\#''} & I \end{array}\right).$$

Hence $\theta^{-1}s\theta = s_\Lambda$ modulo $(H_s \cap H_t)$ and the same holds for $\theta^{-1}t\theta$. Since both s_Λ and t_Λ are the identity on $H_s \cap H_t$, the order of $s_\Lambda\, t_\Lambda$ is equal to that of st. Thus Lemma 12-1 follows from Lemma 6-1.

Corollary 12-1. The mapping $s \longmapsto s_\Lambda \in GL(E^*)$ defines a representation ρ_Λ of W.

Therefore W operates on E^* and the action of $w \in W$ on E^* will be denoted by w_Λ. Since

$$\cos \frac{\pi}{p_{st}} = 0 \text{ for } p_{st} = 2 \text{ and } \lambda_{ss} = 1,$$

the representation $w \longmapsto w_\Lambda$ is determined by the restriction of Λ to pairs $(s,t) \in S \times S$ such that $p_{st} > 2$. This restriction can be regarded as 1 cocycle on the graph of $\mathfrak{M}$ with values in $\mathbb{R}^+$.

<u>Lemma</u> 12-2. <u>For any</u> $w \in W_{\{s,t\}}$ $(s,t \in S,\ s \neq t)$ <u>either</u>

(1) $w_\Lambda(A_s \cap A_t) \subset A_s$ <u>and</u> $\ell(sw) = \ell(w) + 1,$

or

(2) $w_\Lambda(A_s \cap A_t \subset s_\Lambda\ A_s$ <u>and</u> $\ell(sw) = \ell(w) - 1.$

<u>Proof</u>. Let θ be an element of $GL(E^*)$ defined in the proof of Lemma 12-1. Since we have

$$s_\Lambda = \theta^{-1} s \theta \bmod H_s \cap H_t,$$

$$t_\Lambda = \theta^{-1} t \theta \bmod H_s \cap H_t,$$

we have, for any $w \in W_{\{s,t\}}$,

$$w_\Lambda = \theta^{-1} w \theta \bmod H_s \cap H_t.$$

Since θ leaves stable A_s and A_t, we get

$$w_\Lambda(A_s \cap A_t) \subset \theta^{-1} w(A_s \cap A_t) + H_s \cap H_t$$

We know by Lemma 8-1 that either (1) $w_\Lambda(A_s \cap A_t) \subset \theta^{-1} A_s + H_s$ and $\ell(sw) = \ell(w) + 1$, or else (2) $W_\Lambda(A_s \cap A_t) \subset \theta^{-1}(sA_s) + H_s$ and $\ell(sw) = \ell(w) - 1$. Now $\theta^{-1} A_s + H_s = A_s + H_s = \overline{A}_s$ and $\theta^{-1}(sA_s) + H_s = \theta^{-1}(-A_s) + H_s = -A_s + H_s = -A_s = sA_s$ and the lemma is proved.

Similarly we can prove an analogue of Main Lemma in §9 by defining Ω_Λ as the interior of $W_\Lambda\overline{C}$ and replacing Ω and ρ by Ω_Λ and ρ_Λ respectively. The results on Ω obtained so far depend basically only on Main Lemma. Since the analogue of Main Lemma holds for Ω_Λ and ρ_Λ, Ω_Λ has analogous properties as Ω. In particular, Ω_Λ *is a convex cone*.

§13. *Properties of the Representation* ρ_Λ.

Assume now the Coxeter system $\mathcal{W}\mathcal{P}$ = {W,S} is irreducible and $\det(a^{\Lambda}_{st}) \neq 0$.

Lemma 13-1. *(E.B. Vinberg - Katz, [10]).* *If there exists a non-zero symmetric bilinear form invariant under the representation* ρ_Λ, *then* $a^{\Lambda}_{st}\ a^{\Lambda}_{tr}\ a^{\Lambda}_{rs} = a^{\Lambda}_{ts}\ a^{\Lambda}_{sr}\ a^{\Lambda}_{rt}$ *for any* s, t, r $\in$ S.

Proof. Let φ be a bilinear form invariant by ρ_Λ. Then we have $\varphi(s_\Lambda e_s^\Lambda, s_\Lambda e_t^\Lambda) = \varphi(e_s^\Lambda, e_t^\Lambda)$. On the other hand, we have $\varphi(s_\Lambda e_s^\Lambda, s_\Lambda e_t^\Lambda) = \varphi(-e_s^\Lambda, e_t^\Lambda - 2a_{ts}^\Lambda e_s^\Lambda) = -\varphi(e_s^\Lambda, e_t^\Lambda) + 2a_{ts}^\Lambda \varphi(e_s^\Lambda, e_s^\Lambda)$. Therefore $a_{ts}^\Lambda \varphi(e_s^\Lambda, e_s^\Lambda) = \varphi(e_s^\Lambda, e_t^\Lambda)$. From the irreducibility and from $\det(a_{st}^\Lambda) \neq 0$, it follows that $\varphi(e_s^\Lambda, e_s^\Lambda) \neq 0$ for any $s \in S$. Since φ is symmetric, we have

$$\varphi(e_s^\Lambda, e_t^\Lambda)\varphi(e_t^\Lambda, e_r^\Lambda)\varphi(e_r^\Lambda, e_s^\Lambda) = \varphi(e_t^\Lambda, e_s)\varphi(e_s^\Lambda, e_r^\Lambda)\varphi(e_r^\Lambda, e_t^\Lambda),$$

and this proves the lemma.

Now the conditions $a_{st}^\Lambda a_{tr}^\Lambda a_{sr}^\Lambda = a_{ts}^\Lambda a_{sr}^\Lambda a_{rt}^\Lambda$ $(s,t,r \in S)$ are equivalent to

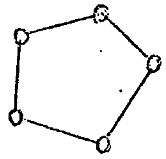
$$(13\text{-}1) \qquad (\lambda_{st} \lambda_{tr} \lambda_{rs} - \lambda_{ts} \lambda_{sr} \lambda_{rt}) \cos\frac{\pi}{p_{st}} \cos\frac{\pi}{p_{tr}} \cos\frac{\pi}{p_{rs}} = 0$$

Remark. Since any finite linear group leaves a symmetric bilinear form invariant, if W is finite, the relations (13-1) hold for any $\Lambda = (\lambda_{st})$. Since we can choose $\lambda_{st}(s,t \in S)$ such that $(\lambda_{st} \lambda_{tr} \lambda_{rt} - \lambda_{ts} \lambda_{sr} \lambda_{rt}) \neq 0$ for any $\{s,t,r\} \subset S$, $\cos\frac{\pi}{p_{st}} \cos\frac{\pi}{p_{tr}} \cos\frac{\pi}{p_{rs}} = 0$ holds for any $\{s,t,r\} \subset S$. This is equivalent to saying that the graph of a finite Coxeter group contains no triangle. More generally:

Proposition 13-1. If W is finite, the graph of the

Coxeter group contains no cycle,

e.g.



Proof. Suppose there exists $s_1, \ldots, s_m$ such that $p_{s_i s_{i+1}} > 2$ $(i = 1, \ldots, m)$ and $p_{s_m s} > 2$. We may assume $m > 3$ and $p_{s_i s_j} = 2$ if $|i - j| \neq 1$ and set $x = \sum_{=1}^{m} e_{s_i}$.

Then we have $B(x,x) = \sum_{=1}^{m} B(e_{s_i}, e_{s_i}) + 2\sum_{i<j} B(e_{s_i}, e_{s_j})$, and since $B(e_{s_i}, e_{s_{i+1}}) \leq \frac{1}{2}$, we have $B(x,x) \leq m + 2(m-1)(-\frac{1}{2}) + 2(-1) < 0$. Since W is finite, by Theorem 7-1 (Witt), B is positive definite.

Remark 13-1. If the graph of a Coxeter group contains no cycle, the linear representation ρ_Λ is similar to the dual canonical representation. More precisely, ρ_Λ is similar to the dual canonical representation ρ^* if and only if Λ, regarded as a 1 cocycle over the graph of W^{ρ}, is cohomologous to 0.

Proposition 13-2. The representation ρ_Λ is irreducible.

Proof. Let F be an invariant subspace. If F contains some e_s^Λ, then F is equal to E*, because the graph is irreducible and $\det(a_{st}^\Lambda) \neq 0$. Now suppose $F \neq \{0\}$. Then for any $x \in F \{0\}$ and $s \in S$, we have $F \quad s_\Lambda x - x = 2\langle e_s, x\rangle e_s^\Lambda$. There exists some $s \in S$ such that $\langle e_s, x\rangle \neq 0$. Then e_s^Λ is in F, which implies F = E*.

Proposition 13-3. *If* W *is infinite, the cone* Ω_Λ *contains no straight line*.

Proof. We define F by $F = \{v \in E^* | \quad x \in E^*, x + \theta v \in \Omega$ for all $\theta \in \mathbb{R}\}$. F is an invariant subspace and, by Proposition 13-2, it is either {0} or E*. If F = E*, we have $\Omega_\Lambda = E^*$ and in this case W is finite by an analogue of Theorem 11-1.

§14. *An Example* (cf. [10]).

Let S_0 be a set consisting of three elements s, t and r. Let P_0 be the following Coxeter matric with indices in S_0;

$$P_0 = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} s & t & r \\ \end{array}$$

$$P_0 = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

We consider a Coxeter system $\mathcal{WP} = \{W, S_0\}$. The matrix $B_0 = (a_{st})$ is as follows:

$$\begin{array}{cc} & \begin{array}{ccc} s & t & r \end{array} \\ B_0 = & \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix} . \end{array}$$

We take as Λ the following matrix Λ_0,

$$\Lambda_0 = \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 2 & 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \sqrt{2} & 1 \end{pmatrix} .$$

Then the matrix $B_0^\Lambda = (a_{st}^\Lambda)$ is as follows,

$$B_0^\Lambda = \begin{pmatrix} 1 & -\frac{1}{4} & -1 \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{4} & -1 & 1 \end{pmatrix} .$$

Then $\det B_0^\Lambda = -\frac{33}{32}$. We can easily check that there is no invariant symmetric bilinear form under W_{Λ_0} (see Lemma 13-1). By Proposition 13-2, Ω_{Λ_0} contains no straight line.

We define the subgroup $G(\Omega_{\Lambda_0})$ of $GL(E^*)$ by $G(\Omega_{\Lambda_0}) = \{\alpha \in GL(E^*) | \alpha(\Omega_{\Lambda_0}) = \Omega_{\Lambda_0}\}$. Then $G(\Omega_{\Lambda_0})$ is not transitive on Ω_{Λ_0}. In fact it is known that a homogeneous cone in $\mathbb{R}^3$ containing no straight line is affinely equivalent to one of the following two homogeneous cones:

(1) $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 > 0, x_2 > 0, x_3 > 0\}$,

(2) $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 - x_3^2 < 0\}$.

Now suppose $G(\Omega_{\Lambda_0})$ is transitive on Ω_{Λ_0}. Since $W_0 (\subset G(\Omega_{\Lambda_0}))$ has no invariant symmetric bilinear form, Ω_{Λ_0} must be simplicial. When Ω_{Λ_0} is simplicial, it is easy to see that any element of finite order in $G(\Omega_{\Lambda_0})$ has order 2 or 3. But this is impossible since the order of tr is 4. Using homotheties it is now possible to define a discrete subgroup in $G(\Omega_{\Lambda_0})$ containing W_Λ such that $W_{\Lambda_0} \backslash \Omega_{\Lambda_0}$ is compact: the convex Ω_{Λ_0} is "quasihomogeneous, but not homogeneous (cf. [1], p.239).

Chapter III

Hyperbolic Coxeter Groups

For this chapter a general reference is Bourbaki [4]. In this chapter we denote by $\mathfrak{W} = \{W,S\}$ a Coxeter system and consider the canonical dual representation of W in $E^* = (\mathbb{R}^S)^*$. Since the representation is faithful we identify W with a subgroup of $GL(E^*)$. For notations, c.f. Chapter I, §7 - §10.

§15. Some Preliminaries.

Lemma 15-1. Let G be a subgroup of $GL(E^*)$ such that G is unimodular and G contains W. Let D be a half-line contained in C and set $G_D = \{g \in G \mid g(D) = D\}$. If $W\backslash G$ has a finite invariant measure, then G_D is compact.

Proof. Denote by μ the Haar measure of G and define an open subset U of G by $\{s \in G \mid s(D) \subset C\}$. Then we have $U \supset G_D$ and $UG_D \overset{\subset}{=} U$. The canonical projection $\pi : G \to W\backslash G$ is injective on U. In fact, suppose there exists $w \in W$ such that $wU \cap U \neq \phi$. Take $s \in U$ such that $ws \in U$. Then $ws(D) \subset C$, which implies $C \cap wC \neq \phi$. Then we have $w = e$ by Theorem 8-1 and this shows that π is injective on U. In particular, $\mu(U) < \infty$. Choose a compact neighborhood K of e in U. We shall now show that there exists a finite subset T in G_D such that for any $g \in G_D$ we have $Kg \cap KT \neq \phi$. In fact, take any subset $\{g_j\}_{j \in J}$ such that $\{Kg_j\}_{j \in J}$ are disjoint. Then the index set J is finite, since $Kg_j \subset UG_D \subset U$, $\mu(Kg) = \mu(K)$ and $0 < \mu(K) < \mu(U) < \infty$. From this observation, we easily see the existence of such a set T. Then $G_D \subset K^{-1}KT$ and since $K^{-1}KT$ is compact, and G_D is closed in $GL(E^*)$, G_D is also compact.

From now on we assume that the canonical bilinear form B is non-degenerate and identify E with E* through B. Then B may be considered as a bilinear form on E*, also denoted by B. Since W leaves B invariant, W can be considered as the discrete subgroup of the orthogonal group O(B).

<u>Theorem</u> 15-1. <u>If the total volume</u> $\mu(W \backslash O(B))$ <u>is finite, the signature of the symmetric bilinear form</u> B <u>is either</u> (n,0) <u>or</u> (n-1, 1).

<u>Proof</u>. For any $x \in C$, we denote by H_x the hyperplane $\{y \in E^* | B(x,y) = 0\}$. Denote by D the halfline $\mathbb{R}^+ x$. Applying Lemma 15-1 to the case $G = O(B)$, we see that G_D is compact, leaves invariant the symmetric bilinear form B_x - the restriction of B to H_x. Since G_D is compact, the bilinear form B_x must be definite. In particular, $B(x,x) \neq 0$ for any $x \in C$. It suffices to show that B_x is positive definite.

$G_D \approx O(B|H_x)$

Suppose it is negative definite. Then $B(x,x)$ would be positive, since otherwise, the symmetric bilinear form B would be negative definite. And this is impossible since $B(e_s^B, e_s^B) = 1$ for any $s \in S$. Therefore the signature of B is $(\overset{+}{1}, \overset{-}{n-1})$. Denote by V the set $\{y \in E^* | B(y,y) > 0\}$. V has two connected components. Since we have $B(e_s^B, e_s^B) = 1$, e_s^B is in V for any $s \in S$. Moreover, if s and t are distinct, e_s^B and e_t^B cannot be in the same connected component of V, since $B(e_s^B, e_t^B) < 0$. Therefore S consists of just two elements. But if this is the case, the canonical bilinear form B is positive definite and we get a contradiction. Hence B_x is positive definite (for any $x \in C$) and the theorem follows.

§16. Hyperbolic Coxeter Groups.

A Coxeter system $\{W,S\}$ is called hyperbolic if the canonical bilinear form B has signature $(n-1, 1)$ and $B(x,x) < 0$ for any $x \in C$. Our aim is to characterize the graphs of hyperbolic Coxeter groups. From now on we assume $\{W,S\}$ to be a hyperbolic Coxeter group.

Denote V by the set $\{y \in E^* | B(y,y) < 0\}$. Then V contains C and V consists of two connected components, which we denote by $V+$ and $V-$, and assume $V+ \supset C$. Then W leaves $V+$

invariant. In fact, for any $s \in S$, we have either $H_{e_s^B} \cap V_+ \neq \phi$ or $H_{e_s^B} \cap V_- \neq \phi$ since $B(e_s^B, e_s^B) = 1$. This means that s has a fixed point either on V_+ or on V_-. On the other hand, since W leaves B invariant, we have either $s(V\pm) = V\pm$ or $s(V\pm) = V\mp$. Therefore $s(V+) = V+$. Denote by q the canonical projection from $E^* - \{0\}$ onto the projective space PE^* associated with E^*. The orthogonal group acts on PE^* and $q(V+)$ is stable under the action of $O(B)$.

Lemma 16-1. $O(B)$ *acts properly on* $q(V+)$.

Proof. From the hyperbolicity, B_x is positive definite for any x in V+. Then the isotropy group of G at any point in $q(V+)$ is compact.

Now we wish to show that $V+ = \Omega$. For that purpose we now prove several lemmas.

Lemma 16-2. V+ *is contained in the simplical cone generated by* $\{-e_s^B\}_{s \in S}$, *i.e. if* $x = \sum_{s \in S} \lambda_s e_s^B$ *is in* V, *then* $\lambda_s \leq 0$ for all s.

Proof. We denote by $\{e_s'\}_{s \in S}$ the basis of E^* dual to the basis $\{e_s\}_{s \in S}$. Then e_s' is contained in $\overline{C}$ and

$\lambda_s = B(x, e'_s)$. Since C is contained in V_+, we have $B(x, e'_s) \leq 0$. Hence $\lambda_s \leq 0$ for any $s \in S$.

Lemma 16-3. $Wx \cap C \neq 0$ *for any* $x \in V_+$.

Proof. Let f be the linear form on E* such that $f(e^B_s) = 1$ for any $s \in S$. Then it is clear from Lemma 16-2 that $f(x) \leq 0$ for any $x \in \overline{V}_+$. For $x \in V_+$, we denote by w_x an element in W such that we have $f(wx) \leq f(w_x x)$ for all $w \in W$. The existence of such an element w_x will be proved by the next lemma. Then we have $w_x x \in \overline{C}$. In fact, put $w_x x = \Sigma \mu_s e'_s$. We have only to show that μ_s is non negative for each $s \in S$. Now $f(tw_x \cdot x) - f(w_x \cdot x) = f(tw_x x - w_x$ $f(-2B(e^B_t, w_x \cdot x)e^B_t) = f(-2\mu_t e^B_t) = -2\mu_t$. From our assumption on w_x, we have $\mu_t \geq 0$.

Lemma 16-4. *Let* f *be the function defined in the proof of Lemma* 16-3. *Then there exists a* w_x *in* W *such that* $f(w_x \cdot x) \geq f(w \cdot x)$ for all $w \in W$.

Proof. We define a real-valued function Φ on V_+ by $\Phi(y) = f(y)^2/B(y \cdot y)$ ($y \in V_+$). Φ is homogeneous and therefore there exists a unique function φ from $q(V_+)$ onto $]-\infty, 0]$ such that $\Phi = \varphi \circ q$. It is easy to check that φ is a proper mapping. In particular the set $\{\xi \in q(V_+) | \varphi(\xi) \geq M\}$ is compact for any M in$]-\infty 0]$. Since G acts properly on $q(V_+)$

(Lemma 16-1) and since W is discrete (Theorem 11-2), $Wq(x)$ is discrete. Therefore there exists a $w_x \in W$ such that $\varphi(w_x\, q(x)) \geq \varphi(wq(x))$ for any $w \in W$, since Φ is proper. Since B is invariant under W, the above implies $f(w_x x) \geq f(wx)$ for any $w \in W$.

Proposition 16-1. $V_+ = \Omega$.

Proof. Since $\Omega \subset W\overline{C}$ $(C \subset V_+)$, and $WV_+ = V_+$, we see that $\Omega \subset W\overline{C} \subset W\overline{V}_+ = \overline{V}_+$ and hence $\Omega \subset V_+$. On the other hand, Lemma 16-3 shows that $V_+ \subset W\overline{C}$ and hence $\Omega = V_+$.

§17. Characterization of Hyperbolic Coxeter Groups.

Theorem 17-1. Let B be the canonical symmetric bilinear form of a hyperbolic Coxeter group and O(B) the orthogonal group of B. Then $W\backslash O(B)$ has finite measure.

Proof. Since O(B) acts properly on $q(\Omega)$, the finiteness of the measure of $W\backslash O(B)$ is equivalent to that of the measure of $W\backslash q(\Omega)$. Now we choose a basis $\{x_0, \ldots, x_{n-1}\}$ of E* ($n = \dim E^*$) such that B is of form: $B(x\cdot x) = -x_0^2 + x_1^2 + \ldots + x_{n-}^2$ The (n-1) form

$$\frac{\sum_i (-1)^i\, x^i\, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n-1}}{(-B(x\cdot x))^{\frac{n}{2}}}$$

induces an $O(B)$ invariant volume element on $q(\Omega)$ (see Proposition 16-1). Since $q(\Omega) = Wq(\overline{C} \cap \Omega)$, it is sufficient to show that the measure of $q(\overline{C} \cap \Omega)$ is finite. Consider the hyperplane $H = \{x_0 = 1\}$ and the integral

$\int_{\overline{C}\cap H} \frac{dx_1 \;\dots\; dx_{n-1}}{\{1-(x_1^2+\dots+x_{n-1}^2)\}^{\frac{n}{2}}}$. The finiteness of this integral implies the finiteness of the measure of $q(\overline{C} \cap \Omega)$. This integral is finite, provided $n > 2$, and n is certainly greater than 2 for hyperbolic Coxeter groups.

Theorem 17-2. *An irreducible Coxeter group is hyperbolic if and only if the following two conditions are verified.*

(a) *for any* $s \in S$, *the symmetric matrix* $(-\cos \frac{\pi}{p_{rt}})_{r,t\in S-s}$ *is positive definite,*

(b) B *is non-degenerate and not positive definite.*

Proof. We first prove the necessity of the conditions (a) and (b). (b) is trivial from the definition of hyperbolic groups. Since $B(x\cdot x) < 0$ for any $x \in C$, we have $B(e_s', e_s') \leq 0$ for any s in S. Define L_s $(s \in S)$ by $\{x \in E^* | B(e_s', x) = 0\}$. Then L_s contains e_t^B for all $t \in S-s$. Since the signature of B is $(n-1, 1)$, B is positive on L_s. The matrix of the restriction of B on L_s is $(-\cos \frac{\pi}{p_{rt}})_{r,t\in S-s}$, which proves (a). Next we show the sufficiency of the conditions. There exists an $x \in E^*$ such that $B(x,x) < 0$ by (b). Set $x = \sum_{s\in S} a_s e_s^B$ and define x_+ (resp. x_-) by $x_+ = \sum_{a_s>0} a_s e_s^B$ (resp. $x_- = \sum_{a_s<0} a_s e_s^B$). Since $B(e_s^B, e_t^B) \leq 0$ for any s,t $(s \neq t)$, we have $B(x_+, x_-) \geq 0$. Hence $B(x,x) \geq B(x_+,x_+) + B(x_-,x_-)$. Therefore we may assume $B(x_+,x_+) < 0$, replacing x by $-x$ if necessary. Let $V = \{x | B(x \cdot x) < 0\}$. Then $x_+ \in V \cap (\sum_{s\in S} \mathbb{R}^+ e_s^B)$. Let V_0 be the connected component of V such that $V_0 \cap (\sum_{s\in S} \mathbb{R}^+ e_s^B) \neq 0$. From (a), we see that $V_0 \cap L_s = \phi$ for any $s \in S$. In particular, we have $V_0 \subset \sum_{s\in S} \mathbb{R}^+ e$ since L_s is on the boundary of $\sum_{s\in S} \mathbb{R}^+ e_s^B$ for any $s \in S$. Now we show that the signature of B is (n-1, 1). Fix an element y in V_0. Suppose there exists a two dimensional subspace, say $\mathbb{R}x + \mathbb{R}y$ $(y \in E^*)$ on which $B < 0$. Then $(\mathbb{R}x + \mathbb{R}y) - \{0\} \subset V_0$ and this is a contradiction, since V_0 is contained in the cone $\Sigma\, \mathbb{R}^+ e_s^B$. Therefore B has the signature (n-1, 1). Next we prove that C is contained in V. Since the signature of B is (n-1, 1) and B is positive on L_s, we have $B(e_s^!, e_s^!) \leq 0$. Hence $e_s^! \in \overline{V}$, which implies $\overline{C} \subset \overline{V}$. Therefore we have $C \subset V$.

A hyperbolic Coxeter group is called a compact Coxeter group if the quotient $W\backslash O(B)$ *is compact*.

Theorem 17-3. *A Coxeter group is compact if and only if the following two conditions are satisfied*.

(a) *for any* $s \in S$, *the matrix* $(-\cos\frac{\pi}{p_{tr}})_{r,t\in S-s}$ *is positive definite*,

(b) B *is non-degenerate and not positive*.

Proof. Proposition 16-1 and Lemma 16-1 show that $O(B)$ acts properly on $q(\Omega)$. Hence the compactness of $W\backslash O(B)$ is equivalent to that of $W\backslash q(\Omega)$. The theorem therefore follows from Theorem 17-2, Theorem 11-3 and Theorem 17-1.

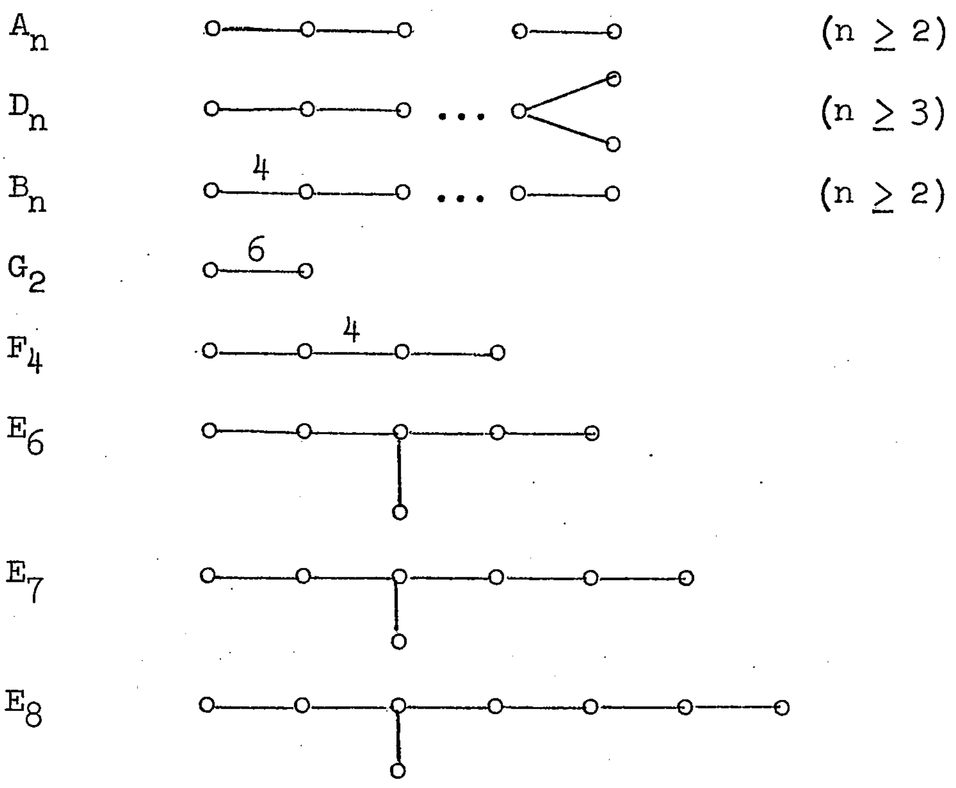
§18. *The List of Hyperbolic Coxeter Groups*.

Lemma 18-1. *If a Coxeter matrix* (p_{st}) *is irreducible and positive, then for any* $s \in S$, *the matrix* $(p_{rt})_{r,t\in S-s}$ *is positive definite*.

Proof. Set $N = \{x \in E | B(x,E) = 0\}$. Note that N is also defined by $N = \{x \in E | B(x,x) = 0\}$. Let x be an element in $N \cap \{e_s' = 0\}$ and let $x = \sum_{t \neq s} a_t e_t$. Since $B(e_t,e_r) \leq 0$ $(t \neq r)$, we have $0 = B(x\cdot x) = \Sigma a_t^2 B(e_t,e_t) + \sum_{t\neq r} a_t a_r B(e_r,e_r)$ $\geq \Sigma|a_t|^2 B(e_t,e_t) + \underset{t\neq r}{} |a_t|\,|a_r|\,B(e_t,e_r) = B(\Sigma|a_t|e_t, \Sigma|a_t|e_t)$ ≥ 0. Therefore $|x| = \Sigma|a_t|e_t$ is also in $N \cap \{e_s' = 0\}$. Set

$T = \{t \in S | a_t \neq 0\}$. Then for any r in $S - T$, we have $0 = B(|x|, e_r) = \Sigma |a_t| B(e_t, e_r)$, which shows that $B(e_t, e_r) = 0$ for any $t \in T$, $r \in S - T$. From the irreducibility we have either $T = \phi$ or $T = S$. Since s is in $S - T$, we have $T = \phi$, i.e. $|x| = 0$ and hence $x = 0$. This completes the proof.

Proposition 18-1. (Coxeter, cf. [5], [11]). The following is the list of the graphs* of all irreducible positive definite Coxeter matrices.



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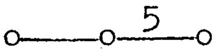
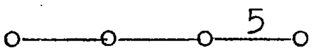
* The number above a 1-simplex $\overline{v_s v_t}$ is p_{st}. Then if p_{st} is 3, we omit it.

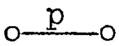
H_3



H_4



D_p



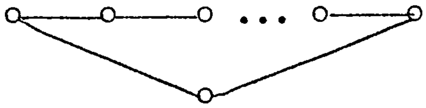
$p \neq 3$
$p \neq 6$

Proof. We will omit the proof of this proposition.

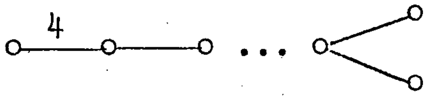
Using Lemma 18-1, we get:

Proposition 18-2. The following is the list of graphs of all irreducible (degenerate) positive Coxeter matrix. ([11]).

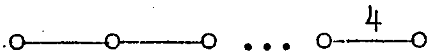
$\tilde{A}_n$



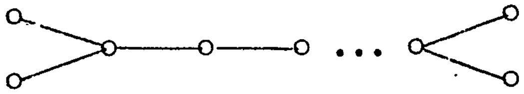
$\tilde{B}_n$



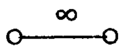
$\tilde{C}_n$



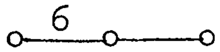
$\tilde{D}_n$



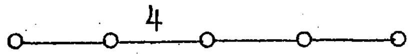
D_∞



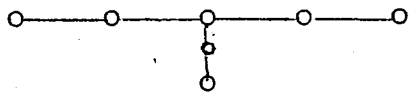
$\tilde{G}_2$



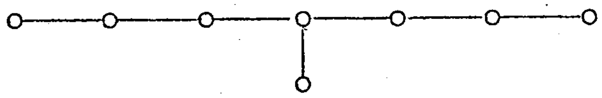
$\tilde{F}_4$



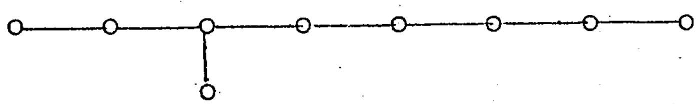
$\tilde{E}_6$



$\tilde{E}_7$



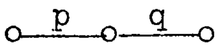
$\tilde{E}_8$



Using Theorem 17-2 and 17-3, the graphs of irreducible hyperbolic Coxeter groups can be constructed from the graphs of positive Coxeter matrices. It appears that *there is no irreducible compact hyperbolic Coxeter group of rank* > 6 *and there is no irreducible hyperbolic Coxeter group of rank* > 10. (Lanner [6], Vinberg [9]).

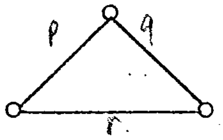
Theorem 18-1. *(Lanner [6]). The graphs of irreducible compact hyperbolic Coxeter groups are the following*:

rank 3



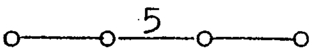
$p = 3 \quad 6 < q < \infty$

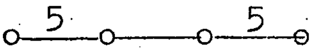
or $3 < p < \infty \quad 4 < q < \infty$

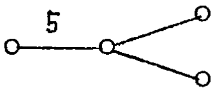


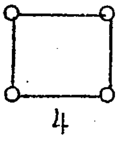
$p + q + r > 9 \qquad p, q, r < \infty$

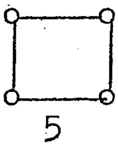
rank 4

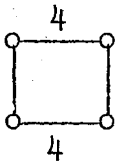


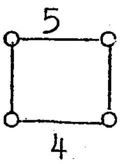


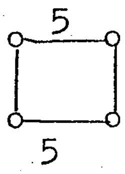




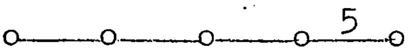


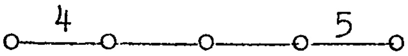


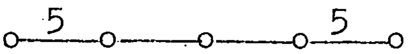


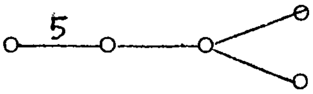


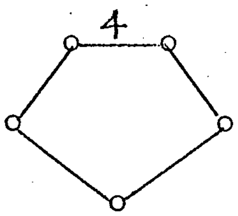
rank 5





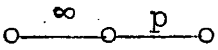




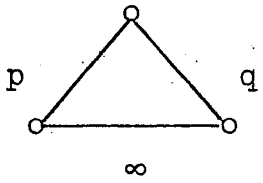


Theorem 18-2. The graphs of irreducible non-compact hyperbolic Coxeter groups are the following (*).

rank 3



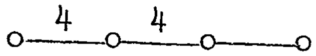
$3 \leq p \leq \infty$

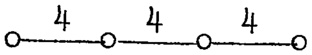


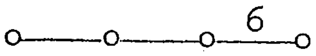
$3 \leq p, q \leq \infty$

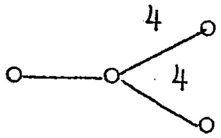
(*) This table was checked by M.M. Chein and N. Spiridon.

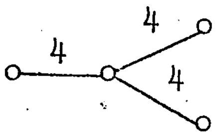
rank 4

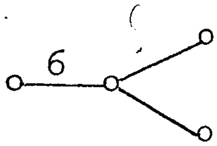


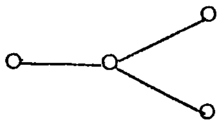


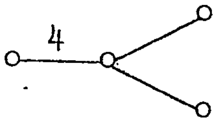


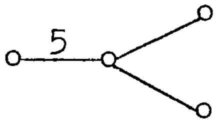


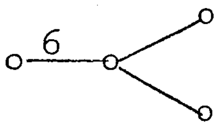


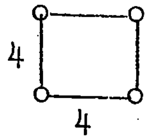


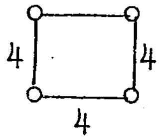


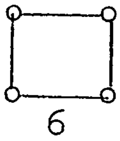


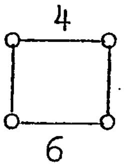


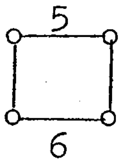


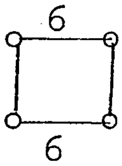


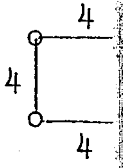


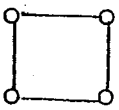


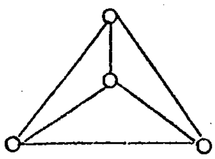












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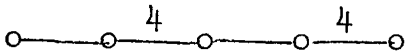

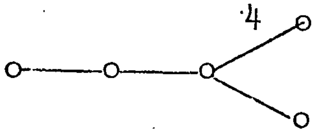

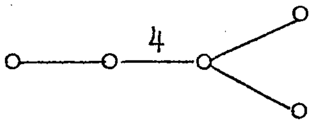

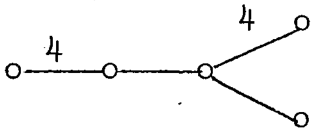

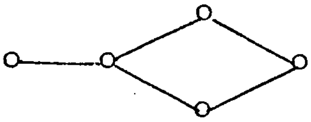

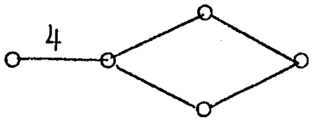

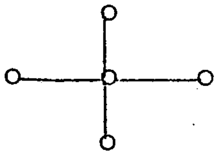

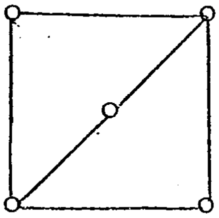

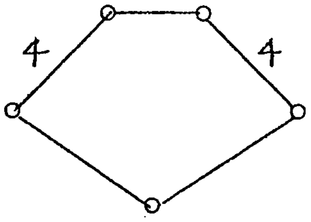

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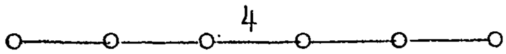
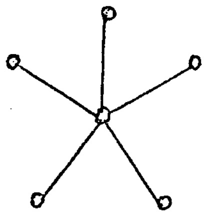
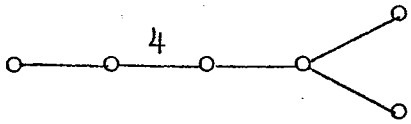
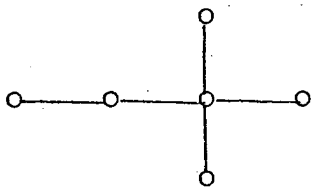
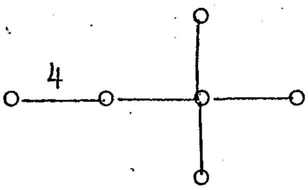
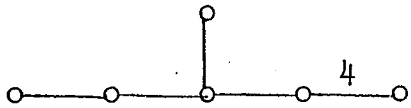


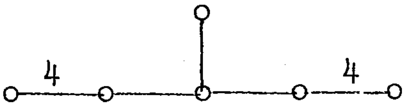


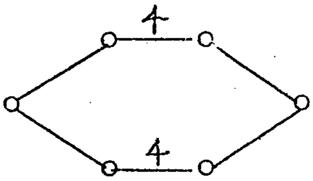


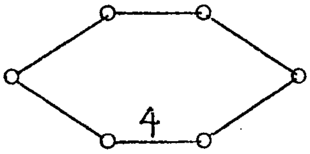


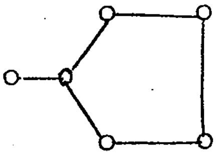




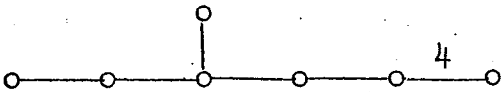


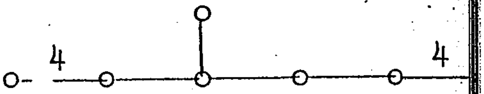


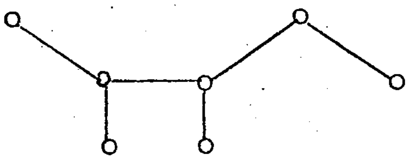


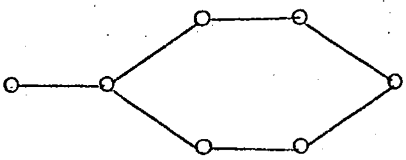


rank 7

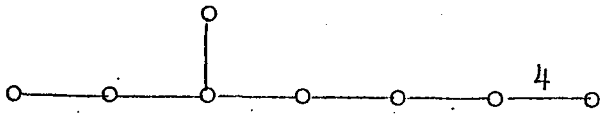


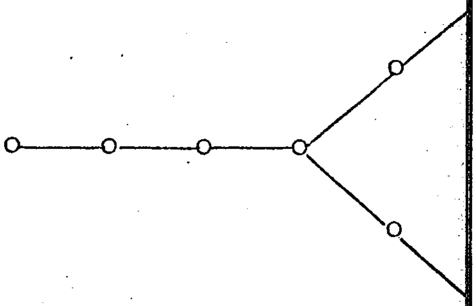


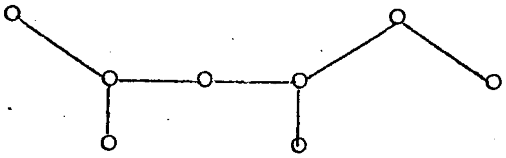


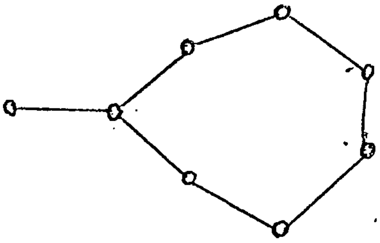


rank 8









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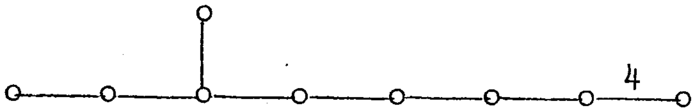

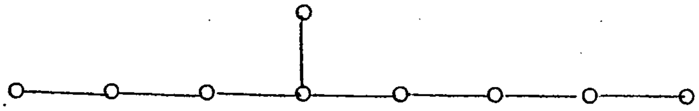

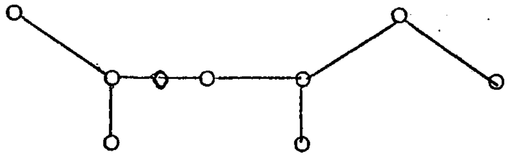

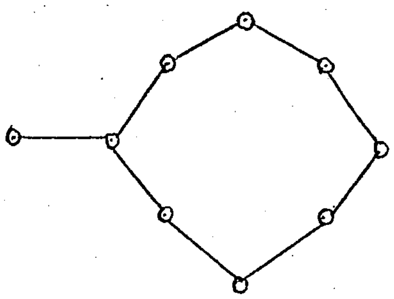

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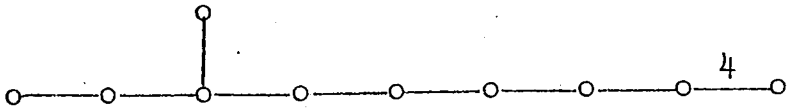
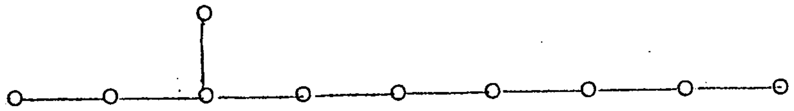
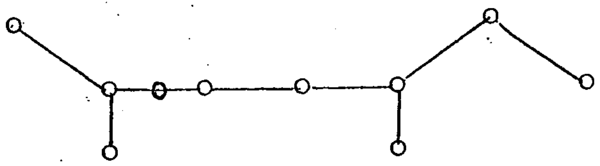






Chapter IV

Arithmetic Hyperbolic Coxeter Groups

§19. Preliminaries from the Algebraic Geometry.

Let X be an algebraic variety in $\mathbb{C}^n$ defined over an algebraic number field k. We denote by $\mathfrak{A}(X)$ the ideal of X in $\mathbb{C}[X_1,\ldots,X_n]$. We put $\mathfrak{A}_k(X) = \mathfrak{A}(X) \cap k[X_1,\ldots,X_n]$. Then $\mathfrak{A}(X)$ is generated by $\mathfrak{A}_k(X)$. Let φ be an isomorphism of k into $\mathbb{C}$ and denote by ${}^{\varphi}X$ the affine algebraic variety defined by the ideal ${}^{\varphi}\mathfrak{A}_k(X) = \{{}^{\varphi}p \mid p \in \mathfrak{A}_k(X)\}$. The variety ${}^{\varphi}X$ is then defined over the field $\varphi(k)$.

For any algebraic number field L, we denote by X_L the set of all L-rational points of X, i.e. $X_L = X \cap L^n$.

Let X (resp. Y) be an algebraic variety in $\mathbb{C}^n$(resp. $\mathbb{C}^m$) defined over an algebraic number field k. Let f be an everywhere defined rational map from X into Y, defined over a field L which is a Galois extension of k. The graph gr(f) of the map f is an affine algebraic variety in C^{n+m} defined over L. For any element σ in the Galois group Gal(L/k), we denote by ${}^{\sigma}f$ the everywhere defined rational map from X (= ${}^{\sigma}X$) into Y (= ${}^{\sigma}Y$) whose graph is given by ${}^{\sigma}gr.(f)$. The map ${}^{\sigma}f$ is also defined over L.

For any L-rational point x in X, we have ${}^{\sigma}(f(x)) = ({}^{\sigma}f)({}^{\sigma}x)$, where $^{\sigma}(x_1,\ldots,x_n) = (^{\sigma}x_1,\ldots,{}^{\sigma}x_n)$. The rational map f is defined over k if and only if $f = {}^{\sigma}f$ for all σ in Gal(L/k).

Lemma 19-1. *Let* X, Y *and* Z *be algebraic varieties defined over an algebraic number field* k. *And let* $f{:}X \to Y$ *and* $g{:}Y \to Z$ *be everywhere defined rational maps defined over a field* L *which is Galois over* k. *Then for any* $\sigma \in \mathrm{Gal}(L/k)$, *we have* $^{\sigma}(g \circ f) = {}^{\sigma}g \circ {}^{\sigma}f$.

Let G be an algebraic subgroup in GL(n,C) defined over a totally real number field k. We denote by $\mathfrak{o}$ the ring of the integers of k. We define the subgroup $G_{\mathfrak{o}}$ of G by $G_{\mathfrak{o}} = \{g = (g^i_j) \in G \mid g^i_j \in \mathfrak{o},\ 1 \leq j \leq n;\ (\det g)^{-1} \in \mathfrak{o}\}$. A subgroup Γ of $G_{\mathbb{R}}$ is, *by the definition, commensurable* with $G_{\mathfrak{o}}$ if $\Gamma \cap G_{\mathfrak{o}}$ is of finite index in both Γ and $G_{\mathfrak{o}}$. A subgroup Γ of $G_{\mathbb{R}}$ is called an *arithmetic subgroup* if the following conditions are satisfied;

(1) for any isomorphism σ of k into $\mathbb{R}$ which is not the identity, $(^{\sigma}G)_{\mathbb{R}}$ is compact;

(2) Γ is commensurable with $G_{\mathfrak{o}}$.

Example 19-1. Let G be an algebraic matrix group in GL(n, $\mathbb{C}$) defined over $\mathbb{Q}$. Then $G_{\mathbb{Z}}$ ($\mathbb{Z} = \mathfrak{o}(\mathbb{Q})$) is arithmetic.

Let G be an algebraic matrix group in GL(n, $\mathbb{C}$) defined over a totally real number field k. If Γ is an arithmetic subgroup of $G_{\mathbb{R}}$, then Γ is discrete in $G_{\mathbb{R}}$. Furthermore, according to Borel and Harish-Chandra [4], G is an irreducible semi-simple group, the quotient $\Gamma \backslash G_{\mathbb{R}}$ is of finite volume.

Let $\mathcal{G}$ be a Lie group. A subgroup W of $\mathcal{G}$ is called an arithmetic subgroup of $\mathcal{G}$ if the following conditions are satisfied:

(1) there exists a totally real algebraic number field k;

(2) there exists an algebraic matrix group G defined over k;

(3) there exists an isomorphism φ of $\mathcal{G}$ onto $G_{\mathbb{R}}$ such that $\varphi(W)$ is an arithmetic subgroup of $G_{\mathbb{R}}$.

For any algebraic matrix group G, we denote by Ad(G) the adjoint group of G and by Aut(G) the group of all automorphisms of G; Ad(G) and Aut(G) are algebraic linear groups.

§20. Arithmetic Hyperbolic Coxeter Groups.

Let $\mathcal{W}$ = {W,X} be a hyperbolic Coxeter group and B its canonical symmetric bilinear form regarded as a form on the space $(\mathbb{R}^S)^* = E^*$. We denote by O(B) and SO(B) the orthogonal group and the special orthogonal group of B respectively. Both are subgroups of GL(E). We have shown

(Cor. 8.1) that W is regarded as a discrete subgroup of O(B) by means of the dual canonical representation.

Let $\Gamma = \mathrm{Ad}(W)$ be the image of W under the adjoint representation of O(B) and define Γ_+ by $\Gamma_+ = \Gamma \cap \mathrm{Ad}(SO(B))$.

The following criterion for Γ_+ to be an arithmetic subgroup of Ad(SO(B)) is a particular case of a criterion proved by Vinberg ([9]).

Theorem 20-1. *Let* $\mathfrak{W} = \{W,S\}$ *be a hyperbolic Coxeter group. Let* p_{st} *be the order of* st *for any* $(s,t) \in S \times S$ *and let* $a_{st} = - \cos (\pi/p_{st})$ *if* $p_{st} < \infty$ *and* $a_{st} = -1$ *if* $p_{st} = \infty$. *Let* k *be the field generated over* $\mathbb{Q}$ *by the elements of the form* $a_{s_1 s_2}\ a_{s_2 s_3} \ldots a_{s_m s_1}$ *and let* $\tilde{k}$ *be the field generated over* $\mathbb{Q}$ *by the elements* $a_{st} (s,t \in S)$.

Then Γ_+ *is an arithmetic subgroup of* Ad(SO(B)) *if and only if, for any isomorphism* τ *of* $\tilde{k}$ *into* $\mathbb{R}$ *which is not identity on* k, *the matrix* $(\tau(a_{st}))$ *positive definite*.

Lemma 20-1. *The volume of* $\Gamma_+ \backslash \mathrm{Ad}(SO(B))$ *if finite*.

Proof. Since the index of SO(B) in O(B) is finite, this lemma follows immediately from Theorem 17-1.

Lemma 20-2. *There exists an algebraic number field* k_0 *and a basis* $\{u_i\}$ *of* E^* *such that*

(1) $B(u_i, u_j) \in \mathbb{Q}$

(2) e_s', e_s^B $(s \in S)$ <u>are in</u> $V_{k_0} = \Sigma\, k_0 \cdot u_i$

(For the definition of e_s^B, cf. §8).

From now on we fix such a field k_0 and a basis $\{u_i\}$ as in Lemma 20-2. We denote by $B_{\mathbb{Q}}$ the restriction of B onto V_{k_0} and by $O(B_{\mathbb{Q}})$ the orthogonal group defined by $B_{\mathbb{Q}}$ and by $SO(B_{\mathbb{Q}})$ the special orthogonal group. (Remark: $O(B_{\mathbb{Q}}) \subset GL(n, \mathbb{C})$ and $SO(B_{\mathbb{Q}})$ is the identity component of $O(B_{\mathbb{Q}})$. We may identify $O(B_{\mathbb{Q}})_{\mathbb{R}}$ with $O(B)$ and $(\mathrm{Ad}\ SO(B_{\mathbb{Q}}))_{\mathbb{R}}$ with $\mathrm{Ad}(SO(B))$.)

<u>Lemma</u> 20-3.

(a) $W \subset O(B_{\mathbb{Q}})_{k_0}$,

(b) $\Gamma_+ \subset \mathrm{Ad}(SO(B_{\mathbb{Q}}))_{k_0}$.

From now on we assume that Γ_+ is an arithmetic subgroup of Ad SO(B). Then there exists a totally real number field k, an algebraic matrix group G defined over k and an isomorphism φ from $G_{\mathbb{R}}$ onto Ad SO(B) such that $\varphi^{-1}(\Gamma_+)$ is arithmetic in $G_{\mathbb{R}}$.

<u>Lemma</u> 20-4. <u>There exists an invariant subgroup</u> Γ_0 <u>of</u> Γ_+ <u>which is of finite index in</u> Γ <u>and is contained in</u> $\varphi(G_k)$.

Proof. $G_{\mathcal{O}} \cap \varphi^{-1}(\Gamma_+)$ is of finite index in $\varphi^{-1}(\Gamma_+)$. Since $G_{\mathcal{O}} \subset G_k$, $G_{\mathcal{O}} \cap \varphi^{-1}(\Gamma_+)$ is of finite index in $\varphi^{-1}(\Gamma_+)$, i.e. $\varphi(G_k) \cap \Gamma_+$ is of finite index in Γ_+. Hence there exists an invariant subgroup Γ_0 of Γ which is of finite index in Γ_+ and is contained in $\varphi(G_k) \cap \Gamma_+$.

Since we identify Ad SO(B) with $(\text{Ad } SO(B_{\mathbb{Q}}))_{\mathbb{R}}$, φ is a map from $G_{\mathbb{R}}$ onto $(\text{Ad } SO(B_{\mathbb{Q}}))_{\mathbb{R}}$. φ induces an isomorphism of the algebraic group G onto the algebraic group $\text{Ad}(SO(B_{\mathbb{Q}}))$ defined over some field L which is a Galois extension of the field K generated by k and k_0.

Lemma 20-5. φ is defined over K.

Proof. It suffices to prove that ${}^{\sigma}\varphi \circ \varphi^{-1}$ is the identity for any $\sigma \in \text{Gal}(L/K)$. Let $u \in \Gamma_0$. Then ${}^{\sigma}u = u$ because Γ_0 is contained in $(\text{Ad } SO(B_{\mathbb{Q}}))_{k_0}$. On the other hand, since $\varphi^{-1}(u)$ is contained in G_k, we have ${}^{\sigma}(\varphi^{-1}(u)) = \varphi^{-1}(u)$ and hence ${}^{\sigma}\varphi \circ \varphi^{-1}(u) = u$ for any $u \in \Gamma_0$. Since Γ_0 is of finite index in Γ_+, the volume of $\Gamma_0 \backslash (\text{Ad } SO(B_{\mathbb{Q}}))_{\mathbb{R}}$ is finite by Lemma 20-1. By a theorem of Borel [2], Γ_0 is Zariski dense in $(\text{Ad } SO(B_{\mathbb{Q}}))_{\mathbb{R}}$. Therefore ${}^{\sigma}\varphi \circ \varphi^{-1}$ is the identity.

Now let L be a Galois extension of k containing K. For each $\sigma \in \text{Gal}(L/k)$, let $A_\sigma = \varphi \circ {}^{\sigma}\varphi^{-1}$. By Lemma 20-5,

A_σ is an automorphism of Ad $SO(B_{\mathbb{Q}})$ defined over L.

Lemma 20-6. $\{A_\sigma\}$ is a 1-cocyle of the group Gal(L/k) with values in $(\mathrm{Aut}(\mathrm{Ad}\ SO(B_{\mathbb{Q}})))_L$.

Proof. We have $A_\sigma {}^\sigma A_\tau = ({\varphi}^\sigma\varphi^{-1})\cdot{}^\sigma({\varphi}^\tau\varphi^{-1}) = {\varphi}^{\sigma\tau}\varphi^{-1} = A_{\sigma\tau}$.

Lemma 20-7. There exists a Galois extension L_0 of k containing K satisfying the following conditions;

(1) there exists a 1-cocyle $\{C_\sigma\}$ ($\sigma \in \mathrm{Gal}(L_0/k)$) with values in $(O(B_{\mathbb{Q}}))_L$;

(2) $A_\sigma = \mathrm{Inn}(\mathrm{Ad}\ C_\sigma)$.

(3) $C_\sigma {}^\sigma Z C_\sigma^{-1} = Z$ for any Z in $S \subset W$.

Proof. There exists a Galois extension L_0 of k containing K such that $AD:(O(B_{\mathbb{Q}}))_{L_0} \to (\mathrm{Ad}\ O(B_{\mathbb{Q}}))_{L_0}$ is surjective. For any $\gamma \in \Gamma$, the automorphism $\mathrm{Inn}(\gamma)$ of $\mathrm{Ad}(SO(B_{\mathbb{Q}}))$ is defined over L_0. Now we assert

$$A_\sigma {}^\sigma(\mathrm{inn}(\gamma))A_\sigma^{-1} = \mathrm{Inn}(\gamma)$$

for any $\sigma \in \mathrm{Gal}(L_0/k)$. In fact, since $\varphi^{-1}(\mathrm{Inn}(\gamma))\varphi\,.\,\varphi^{-1}(\Gamma_0) \subset \varphi^{-1}(\Gamma_0)$ and $\varphi^{-1}(\Gamma_0) \subset G_k$, we have ${}^\sigma(\varphi^{-1}(\mathrm{Inn}(\gamma))\varphi) = \varphi^{-1}\mathrm{Inn}(\gamma)\varphi$ on Γ_0. Since Γ_0 is Zariski-dense, this yields our assertion.

Next we shall show that the automorphism A_σ of $\mathrm{Ad}(SO(B_{\mathbb{Q}}))$ is the restriction of an automorphism of $\mathrm{Ad}(O(B_{\mathbb{Q}}))$ defined over L_0. In fact, take any element $s_0 \in S \subset W$ and set $\gamma = \mathrm{Ad}s_0$. Then γ is contained in $(\mathrm{Ad}\ O(B_{\mathbb{Q}}))_{L_0}$. We denote by ℓ_γ the everywhere regular map of $\mathrm{Ad}(O(B\ ))$ onto itself defined by the left multiplication by γ. Then ℓ_γ is defined over L and interchanges the irreducible components of $\mathrm{Ad}(O(B_{\mathbb{Q}}))$. Let $\mathrm{Ad}^{+}(SO(B_{\mathbb{Q}}))$ and $\mathrm{Ad}^{-}(O(B_{\mathbb{Q}}))$ be the irreducible components of $\mathrm{Ad}(O(B_{\mathbb{Q}}))$. We define A_σ on $\mathrm{Ad}^{-}(O(B_{\mathbb{Q}}))$ by the formula

$$A_\sigma(x) = (\ell_\gamma\, A_\sigma\, {}^{\sigma}\ell_\gamma)(x)$$

for any $x \in \mathrm{Ad}^{-}(O(B_{\mathbb{Q}}))$. Then it is easy to check that

(i) A_σ is an automorphism of $\mathrm{Ad}(O(B_{\mathbb{Q}}))$ defined over L_0,

(ii) $A_\sigma {}^{\sigma}A_\tau = A_{\sigma\tau}$ for any σ, $\tau \in \mathrm{Gal}(L_0, k)$.

Now we know that any automorphism of the algebraic group $\mathrm{Ad}(O(B_{\mathbb{Q}}))$ is an inner automorphism. Since $\mathrm{Ad}(O(B_{\mathbb{Q}}))$ has no center, there exists a unique element $\mathfrak{a}_\sigma$ in $(\mathrm{Ad}(O(B_{\mathbb{Q}})))_{L_0}$ such that $A_\sigma = \mathrm{Inn}(\mathfrak{a}_\sigma)$. Let C_σ be an element of $(O(B_{\mathbb{Q}}))_{L_0}$ such that $\mathrm{Ad}(C_\sigma) = \mathfrak{a}_\sigma$.

Now put $u_s = \mathrm{Inn}(\mathrm{Ad}(s))$ for any $s \in S$. We know that $A_\sigma {}^{\sigma}(us) A_\sigma^{-1} = u_s$ and hence $C_\sigma {}^{\sigma}s\ C_\sigma^{-1} = \pm s$. Since Trace(s) = Trace $({}^{\sigma}s)$, we have

$$C_\sigma {}^{\sigma}s\ C_\sigma^{-1} = s. \tag{20-1}$$

Therefore $(C_\sigma {}^\sigma s\, C_\sigma^{-1})(e_s^B) = s(e_s^B) = -e_s^B$, i.e. $e_s(C_\sigma^{-1} e_s^B) = -C_\sigma^{-1} e_s^B$. Hence $s\, {}^{\sigma^{-1}}(C_\sigma^{-1} e_s^B) = -{}^{\sigma^{-1}}(C_\sigma e_s^B)$. Thus $C_\sigma^{-1} e_s^B = \varepsilon_s e_s^B$ ($\varepsilon_s = \pm 1$) for any s in S. Choose C_σ such that $C_\sigma^{-1} e_{s_0}^B = e_{s_0}^B$. Then $C_\sigma {}^\sigma C_\tau = C_{\sigma\tau}$. Then the assertion (3) follows from (20-1). We have thus proved Lemma 20-7.

From now on we fix the field L_0 as in Lemma 20-7. The following is well known (cf. J.P. Serre. Corps Locaux, Hermann, p. 159).

Lemma 20-8. *Let* $GL(V_0)$ *be the group of automorphisms of the* L_0*-vector space* V_0. *Then* $H^1(\mathrm{Gal}(L_0/k), GL(V_0)) = 0$.

Since $\mathcal{O}(B_{\mathbb{Q}})_{L_0}$ is contained in $GL(V_0)_{L_0}$, there exists an element $\alpha \in GL(V_0)_{L_0}$ such that $C_\sigma = \alpha \cdot {}^\sigma\alpha^{-1}$ for any $\sigma \in \mathrm{Gal}(L_0/k)$. We define a k-vector space V_k by $V_k = k \cdot \alpha(V_0)$. Then $L_0 \cdot V_k = L_0 \cdot V_0$. We denote by B_k the restriction of B to V_k.

Lemma 20-9. $B_k(V_k, V_k)$ *is contained in* k.

Proof. For any $x, y \in V_0$, $B(x,y)$ is in $\mathbb{Q}$. Therefore for any $\sigma \in \mathrm{Gal}(L_0/k)$, we have ${}^\sigma(B(\alpha x, \alpha y)) = B({}^\sigma\alpha {}^\sigma x, {}^\sigma\alpha {}^\sigma y) = B({}^\sigma\alpha \cdot x, {}^\sigma\alpha \cdot y) = B(C_\sigma^{-1} \alpha x, C_\sigma^{-1} \alpha y) = B(\alpha x, \alpha y)$, because C_σ is in $(\mathcal{O}(B_{\mathbb{Q}}))_{L_0}$. Therefore $B(\alpha x, \alpha y)$ is in k.

Denote by $\mathcal{O}(B_k)$ the algebraic orthogonal matrix group defined by B_k. Then $\mathcal{O}(B_k)$ is defined over k. There is a natural isomorphism θ from $\mathcal{O}(B_{\mathbb{Q}})$ onto $\mathcal{O}(B_k)$ defined over L_0, i.e. $\theta(x) = \alpha^{-1}x\alpha$ for any $x \in \mathcal{O}(B_{\mathbb{Q}})$.

Lemma 20-10. *For any* $x \in \mathcal{O}(B_{\mathbb{Q}})_{L_0}$, *we have* ${}^{\sigma}(\theta(\beta)) = \theta(\mathrm{Inn}(\underset{\alpha}{\mathbb{Q}})^{\sigma}x)$.

Proof. ${}^{\sigma}(\theta(x)) = {}^{\sigma}(\alpha^{-1}x\alpha) = {}^{\sigma}\alpha^{-1}\ {}^{\sigma}x{}^{\sigma}\alpha = \alpha^{-1}C_{\sigma}{}^{\sigma}x\ C_{\sigma}^{-1}\alpha = \theta(C_{\alpha}^{\sigma}x\ C_{\alpha}^{-1})$.

Lemma 20-11. W *is contained in* $(\mathcal{O}(B_k))_k$. This follows immediately from (3) of Lemma 20-7.

Lemma 20-12. *The isomorphism* $\psi\ \ \varphi$ *of* G *onto* $\mathrm{Ad}(SO(B_k))$ *is defined over* k, *where* ψ *is the natural map induced by* θ.

Proof. Since $\psi \circ \varphi$ is defined over L_0, it suffices to show ${}^{\sigma}(\psi \circ \varphi) = \psi\ \ \varphi$ for any $\sigma \in \mathrm{Gal}(L_0/k)$, i.e. $\varphi \circ {}^{\sigma}\varphi^{-1} = \psi^{-1} \circ {}^{\sigma}\psi$. For any $w \in W \cap \varphi(G_k)$, we have $({}^{\sigma}\theta)(w) = \theta(\mathrm{Inn}\ C_{\sigma})(w)$. Hence $\psi^{-1} \circ {}^{\sigma}\psi = \mathrm{Inn}(\mathrm{Ad}\ C_{\sigma})\ \gamma = \mathrm{Int}(\alpha C_{\sigma})$, which shows that ${}^{\sigma}\psi\psi^{-1} = \mathrm{Int}(\mathrm{Ad}(\alpha C_{\sigma}))$. On the other hand we have $(\varphi \cdot {}^{\sigma}\varphi^{-1})(\gamma) = A\sigma(\gamma) = \mathrm{Inn}(\mathrm{Ad}(C_{\sigma}) \cdot \gamma$. Since Γ_0 is dense, this yields $\varphi \circ {}^{\sigma}\varphi^{-1} = \psi^{-1} \circ {}^{\sigma}\psi$.

Lemma 20-13. *If* μ *is an isomrophism of* k *into* $\mathbb{R}$ *which is not the identity, then* ${}^{\mu}B_k$ *is either positive definite or* *negative definite.*

Proof. Since $\mathrm{Ad}(SO(B_k))$ is isomorphic to G over k, $({}^{\mu}\mathrm{Ad}(SO(B_k)))_{\mathbb{R}}$ is compact, which implies that $({}^{\mu}\mathcal{O}(B_k))_{\mathbb{R}}$ is compact. Hence B_k must be either positive definite or negative definite.

Lemma 20-14. *For any* $s \in S$, *there exists a real number* λ_s *such that* $\lambda_s e_s^B$ *is contained in* V_k.

Proof. Since $\mathbb{R} \cdot V_k = E^*$, there exists an $x \in V_k$ such that $B(x, e_s^B) \neq 0$. Since W is contained in $(\mathcal{O}(B_k))_k$ (Lemma 20-11), $s(x) - x = 2B(x \cdot e_s^B)e_s^B \in V_k$.

We fix once and for all a λ_s for each $s \in S$.

Lemma 20-15. *For each* $s \in S$, λ_s^2 *is in* k.

Proof. We know that $B(V_k\ V_k) \in k$, and thus $\lambda_s^2 = B(\lambda_s e_s^B, \lambda_s e_s^B) \in k$.

Lemma 20-16. *For any subset* $\{s_1, \ldots, s_m\}$ *of* S, $a_{s_1 s_2}\, a_{s_2 s_3} \cdots a_{s_m s_1}$ *is in* k, *where* $a_{s_i s_j} = -\cos(\pi/p_{s_i s_j})$.

Proof. Since $\lambda_s \lambda_t\, B(e_s^B\ e_t^B)$ is in k, $\lambda_{s_1}\lambda_{s_2}\lambda_{s_2} \cdots \lambda_{s_m}\lambda_{s_1}\lambda$

$a_{s_1 s_2} \cdots a_{s_m s_1}$ is in k. From Lemma 20-15 it follows that

$a_{s_1 s_2} \cdots a_{s_m s_1} \in k$.

Lemma 20-17. *Let* $\tilde{K} = K(\{\lambda_s\}_{s \in S})$. *For any isomorphism*

τ *of* $\widetilde{K}$ *into* $\mathbb{C}$ *such that the restriction of* τ *to* k *is not the identity,* $({}^{\tau}a_{st})$ *is a positive definite real symmetric matrix.*

Proof. Let $B(\lambda_s e_s^B, \lambda_t e_t^B) = b_{st}$ $(\in k)$. Then we have ${}^{\tau}b_{st} = {}^{\tau}\lambda_s \; {}^{\tau}\lambda_t \; {}^{\tau}a_{st}$. Since $a_{ss} = 1$ for any $s \in S$, we have ${}^{\tau}b_{ss} = ({}^{\tau}\lambda_s)^2$. Since k is a totally real, ${}^{\tau}b_{ss}$ is in $\mathbb{R}$. Hence ${}^{\tau}\lambda_s$ is either real or purely imaginary. On the other hand $({}^{\tau}b_{st})$ is definite from the Lemma 20-13, hence all of ${}^{\tau}\lambda_s$ $(s \in S)$ are either real or purely imaginary according as $({}^{\tau}b_{st}) > 0$ or $({}^{\tau}b_{st}) < 0$. Then the matrix $({}^{\tau}a_{st})$ is real and definite, and since $a_{ss} = 1$, ${}^{\tau}(a_{st})$ is positive definite.

Lemma 20-18. The field k is generated over $\mathbb{Q}$ by $\{a_{s_1 s_2} \, a_{s_2 s_1} \ldots a_{s_m} \, a_{s_1}\}$.

Proof. Denote by k' the field $\mathbb{Q}(\{a_{s_1 s_2} \ldots a_{s_m s_1}\})$. By Lemma 20-16, k' is a subfield of k. Suppose there exists an element ν in k such that $\nu \notin k'$. Then there exists an isomorphism τ of k into   such that $\tau|k'$ is the identity and $\tau\nu \neq \nu$. We can extend τ to an isomorphism of k. By Lemma 20-17, ${}^{\tau}(a_{st})$ is positive definite. On the other hand, the determinant of any minors of the matrix (a_{st}) is in k', so that the determinat of any minor of the matrix ${}^{\tau}(a_{st})$ is also in k'. Hence (a_{st}) must be positive definite, which is a contradiction.

Proof of Theorem 20-1. The sufficiency of the condition in Theorem 20-1 follows easily from Lemma 20-17 and 20-18. (We omit the proof of the necessity.)

Proposition 20-1. *If a hyperbolic Coxeter group* $\mathfrak{W} = \{W\ S\}$ *is not compact and if* Γ_+ *is arithmetic, then* $k = \mathbb{Q}$.

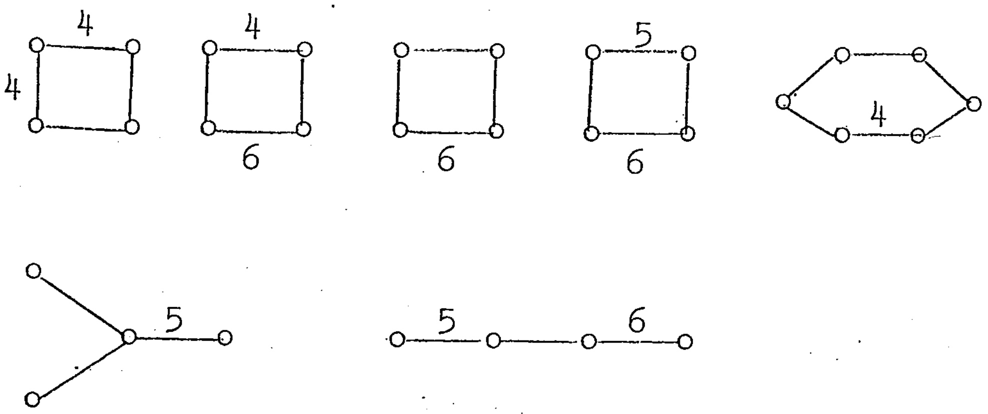
Proof. From Theorem 17-2 and Theorem 17-3, there exists s in S such that $(b_{rt})_{r,t \neq s}$ is degenerate and positive. Suppose $k \neq \mathbb{Q}$. Then there exists an isomorphism τ of k into $\mathbb{R}$ different from the identity. We may extend τ to an isomorphism of k. By Lemma 20-17, ${}^{\tau}(a_{st})$ is positive definite. Hence ${}^{\tau}(a_{rt})_{r,t \neq s}$ is also positive definite. Then $(a_{rt})_{r,t \neq s}$ is non-degenerate and this is a contradiction

Remark: This result follows also directly from the general "Godemaut criterion", cf. [3] Theorem 11.6.

§21. Examples of non-arithmetic subgroups.

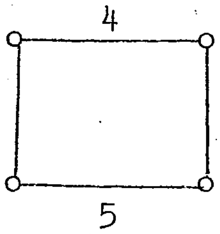
In view of Lemma 20-18 we are interested in hyperbolic Coxeter groups whose graphs contain at least one edge with index > 3.

Example 21-1. Non-compact case. We consider the non-compact hyperbolic Coxeter groups whose graphs are respectivel



In these cases k is respectively $Q(\sqrt{2})$, $Q(\sqrt{2}, \sqrt{3})$, $Q(\sqrt{3})$, $Q(\sqrt{3}, \sqrt{5})$, $Q(\sqrt{2})$, $Q(\sqrt{5})$, $Q(\sqrt{5})$. Therefore, in each case the corresponding group Γ_+ is not arithmetic (cf. Prop. 20-1).

Example 21-2. Compact case. Consider the compact hyperbolic Coxeter group with the graph



Here we have $k = \tilde{k} = Q(\sqrt{2}, \sqrt{5})$. Let τ be the isomorphism of k into defined by $\tau(\sqrt{5}) = -\sqrt{5}$, $\tau(\sqrt{2}) = \sqrt{2}$. Then ${}^{\tau}(a_{st})$ is not positive definite and hence the corresponding group Γ_+ is not arithmetic by Theorem 20-1.

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